**Integer Points in Polyhedra**

Spring 2009

Assignment Sheet 3

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**Exercise 1 (Well-ordered basis)**
Let \( b_1, b_2 \in \mathbb{Z}^2 \) be a basis of two-dimensional lattice. Show that we can transform this basis, in constant time, into a well-ordered basis, i.e., basis \( b_1', b_2' \) satisfying

\[
\| b_1' \| \leq \| b_2' - b_1' \| \leq \| b_2' \| \leq \| b_2' + b_1' \|.
\]

**Exercise 2 (Running time of the generalized Gauss algorithm)**
Prove that the generalized Gauss reduction algorithm in \( \mathbb{R}^2 \) runs in polynomial time.

**Exercise 3 (Closest vector in \( \mathbb{R}^2 \))**
Describe an efficient algorithm that, provided a lattice \( \Lambda \subseteq \mathbb{R}^2 \) and a vector \( v \in \mathbb{R}^2 \), finds a lattice vector \( z \in \Lambda \) with \( \| z - v \| \) as small as possible.

**Exercise 4 (Integral points in fundamental parallelepiped)**
Let \( b_1, b_2, \ldots, b_n \) be linearly independent integral vectors in \( \mathbb{R}^n \). Show that the fundamental parallelepiped

\[
P(b_1, b_2, \ldots, b_n) = \left\{ \sum_{i=1}^{n} \lambda_i b_i : 0 \leq \lambda_i < 1, \ i = 1,2,\ldots,n \right\}
\]

contains exactly \( \det(b_1, b_2, \ldots, b_n) \) integral vectors.

**Exercise 5 (Integral points in ellipsoids)**
Given an algorithm computing the shortest lattice vector wrt \( \| \cdot \|_2 \), describe an algorithm to find an integral point in a given ellipsoid \( \| Ax \|_2 \leq b \).