Exercises

Approximation Algorithms

Spring 2010

Sheet 2

Note: This is just one way, a solution could look like. We do not guarantee correctness. It is your task to find and report mistakes.

Exercise 1
Give a family of instances, where Christophides algorithm for TSP gives a solution whose approximation guarantee indeed tends to \( \frac{3}{2} \).

Solution:
Consider a graph with nodes \( u_1, \ldots, u_n \) on the upper layer, \( v_1, \ldots, v_n \) on the lower layer. Pairs \( v_i, u_i \) and consecutive \( v_i \)'s, \( u_i \)'s are connected by unit cost edges. Other edges have shortest path distances.

An MST \( T \) of cost \( 2n - 1 \) is as follows.

A matching on both odd degree nodes costs \( n + 1 \). The cheapest tour costs \( 2n \).

Exercise 2
For a parameter \( k \in \mathbb{N} \), we consider the following SET COVER instance: Choose elements \( U := \mathbb{Z}_2^k \setminus \{(0, \ldots, 0)\} \). For each vector \( z \in \mathbb{Z}_2^k \), we define a set \( S_z := \{ y \in U \mid z \cdot y \equiv 2 \mod 2 \} \) where \( z \cdot y \equiv 2 \sum_{i=1}^{k} z_i y_i \) is the standard scalar product mod 2. Hence we have \( n := |U| = 2^k - 1 \) elements and \( 2^k \) sets. All sets have unit cost.

Example: For \( k = 2 \) we have elements \( U = \{(1,0),(0,1),(1,1)\} \) and sets \( S_{(0,0)} = \emptyset, S_{(0,1)} = \{(0,1),(1,1)\}, S_{(1,0)} = \{(1,0),(1,1)\}, S_{(1,1)} = \{(1,0),(0,1)\} \).

Show that \( OPT \geq k \) and \( OPT_f \leq 2 \) (hence the integrality gap is \( \Omega(\log n) \)).
Solution:

We claim that every element is in \( \frac{1}{2} 2^k \geq \frac{k}{2} \) many sets (i.e. \( \frac{1}{2} \) of the sets): Let’s fix a \( y \in U \). Say \( y_i = 1 \) (\( y \neq (0, \ldots, 0) \)). Fix any choice of \( z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_k \in \{0, 1\} \) there is exactly one choice for \( z_i \) s.t. \( z \cdot y \equiv 2 \mod 2 \). Hence if we choose \( x_i := \frac{2^k}{n} \), then each element is covered fractionally at least once (in solution \( x \) ), thus \( OPT_f \leq n \cdot \frac{k}{2} = 2 \).

Next suppose for contradiction that \( k-1 \) sets \( S_i, \ldots, S_{i-1} \) suffice to cover all elements. Consider the 0/1 matrix \( A \) with rows \( z^1, \ldots, z^{k-1} \). The rank of this matrix w.r.t. \( \mathbb{Z}_2 \) can be at most \( k-1 \). Hence there must be a non-zero vector \( y \in ker(A) \), i.e. \( y \cdot z_i \equiv 0 \) for \( i = 1, \ldots, k-1 \). Hence \( y \notin S_\phi \).

Exercise 3

The Set Packing problem is as follows: Given a family of sets \( S_1, \ldots, S_m \subseteq U \) of cardinality \(|S_i| = 3 \) with profits \( c(S_i) \), find a subset of these sets that maximizes the profit, while each element is covered at most once. Consider a straightforward integer linear programming formulation

\[
\begin{align*}
\max & \sum_{i=1}^m c(S_i) \cdot x_i \quad (ILP) \\
\text{s.t.} & \sum_{i : j \in S_i} x_i \leq 1 \quad \forall j \in U \\
& x_i \in \{0, 1\} \quad \forall i
\end{align*}
\]

where \( x_i \) indicates, whether to take set \( S_i \). Let \( OPT \) be its optimum value and \( OPT_f \) be the optimum value of its fractional relaxation. Prove that \( \frac{OPT_f}{OPT} \leq O(1) \) (for a big enough constant).

**Hint:** A suitable randomized rounding should do the job.

Solution:

Compute an fractional solution \( x^* \in [0, 1]^m \) of value \( OPT_f \). Then perform the following rounding algorithm:

1. Choose set \( S_i \) with probability \( \frac{1}{6} x^*_i \)

2. Consider all elements \( j \in U \): If \( j \) is covered by more than 1 set, remove all sets containing \( j \) from the solution

Let \( I_1 \subseteq \{1, \ldots, m\} \) be the sets chosen in (1) and \( I_2 \) be the sets chosen in (1) and surviving (2). Consider a set \( S_i \):

\[
\Pr[i \in I_2] = \Pr[\bigcap_{j \in S_i \cap S_j \neq i} j \notin I_1] = \frac{1}{6} x^*_i \cdot \Pr[\bigcup_{j : j \notin I_1 \cup j \in I_1} j \in I_1] \geq \frac{1}{12} x^*_i
\]

Using that

\[
\Pr[\bigcup_{j : j \notin I_1 \cup j \in I_1} j \in I_1] \leq \sum_{j \notin I_1 \cup j \in I_1} \Pr[j \in I_1] = \frac{1}{6} \sum_{j \notin I_1 \cup j \in I_1} x^*_j \leq \frac{1}{2}
\]

Hence the solution \( I \) has an expected profit of

\[
\sum_{i=1}^m \Pr[i \in I] \cdot c(S_i) \geq \frac{1}{12} \sum_{i=1}^m x^*_i c(S_i) = \frac{1}{12} OPT_f
\]