Chapter 5
Integer Programming

An *integer program* is a problem of the form

$$\begin{aligned} \max c^T x \\ Ax \leqslant b \\ x \in \mathbb{Z}^n, \end{aligned}$$

where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$.

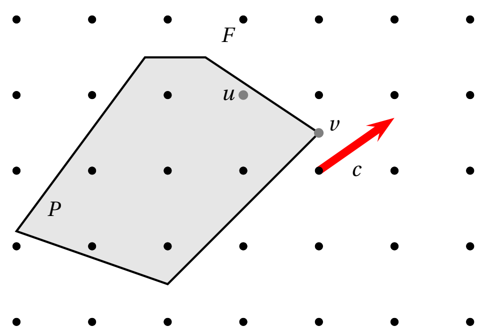

Fig. 5.1: This picture illustrates a polyhedron P an objective function vector c and optimal points u, v of the integer program and the relaxation respectively.

The difference to linear programming is the *integrality constraint* $x \in \mathbb{Z}^n$. This powerful constraint allows to model discrete choices but, at the same time, makes an integer program much more difficult to solve than a linear program. In fact one can show that integer programming is NP-hard, which means that it is *in theory* computationally intractable. However, integer programming has nowadays become an important tool to solve difficult industrial optimization problems efficiently. In this chapter, we characterize some integer programs which are easy to solve, since the *linear programming relaxation* $\max\{c^T x\colon Ax \leqslant b\}$ yields already an optimal integer solution. The following observation is crucial.

Theorem 5.1. *Suppose that x^* is an integral optimum solution of the linear programming relaxation* $\max\{c^T x\colon Ax \leqslant b\}$ *is integral, i.e., $x^* \in \mathbb{Z}^n$, then x^* is also an optimal solution to the integer programming problem* $\max\{c^T x\colon Ax \leqslant b, x \in \mathbb{Z}^n\}$

Before we present an example for the power of integer programming we recall the definition of an undirected graph.

Definition 5.1 (Undirected graph, matching). An *undirected graph* is a tuple $G = (V, E)$ where V is a finite set, called the *vertices* and $E \subseteq \binom{V}{2}$ is the set of *edges* of G. A *matching* of G is a subset $M \subseteq E$ such that for all $e_1 \neq e_2 \in M$ one has $e_1 \cap e_2 = \emptyset$.

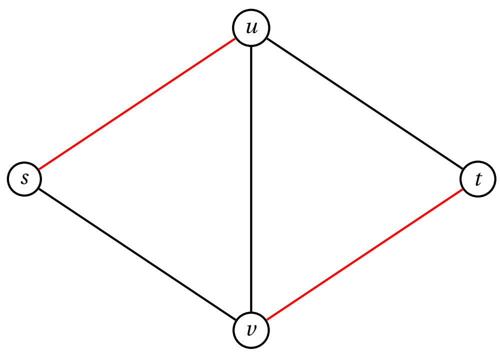

Fig. 5.2: A graph with 4 nodes $V = \{s, u, v, t\}$ and 5 edges $E = \{\{s, u\}, \{s, v\}, \{u, v\}, \{u, t\}, \{v, t\}\}$. The red edges are a matching of the graph

We are interested in the solution of the following problem, which is called *maximum weight matching* problem. Given a graph $G = (V, E)$ and a weight function $w : E \to \mathbb{R}$, compute a matching with maximum weight $w(M) = \sum_{e \in M} w(e)$.

For a vertex $v \in V$, the set $\delta(v) = \{e \in E\colon v \in e\}$ denotes the *incident* edges to v. The maximum weight matching problem can now be modeled as an integer program as follows.

$$\begin{array}{c} \max \sum_{e \in E} w(e) x(e) \\ v \in V : \sum_{e \in \delta(v)} x(e) \leqslant 1 \\ e \in E : x(e) \geqslant 0 \\ x \in \mathbb{Z}^{|E|}. \end{array} \tag{5.1}$$

Clearly, if an integer vector $x \in \mathbb{Z}^n$ satisfies the constraints above, then this vector is the *incidence vector* of a matching of G. In other words, the integral solutions to the constraints above are the vectors $\{\chi^M\colon M \text{ matching of } G\}$, where $\chi^M(e) = 1$ if $e \in M$ and $\chi^M(e) = 0$ otherwise.

5.1 Integral Polyhedra

In this section we derive sufficient conditions on an integer program to be solved easily by an algorithm for linear programming. A central notion is the one of an integral polyhedron.

Definition 5.2 (Valid inequality, face, vertex). Let $P = \{x \in \mathbb{R}^n\colon\ Ax \leqslant b\}$ be a polyhedron. An inequality $c^T x \leqslant \beta$ is *valid* for P if $c^T x^* \leqslant \beta$ for all $x^* \in P$. A *face* of P is a set of the form $P \cap \{x \in \mathbb{R}^n\colon\ c^T x = \beta\}$ for a valid inequality $c^T x \leqslant \beta$ of P. If a face consist of one point, then it is called a *vertex* of P.

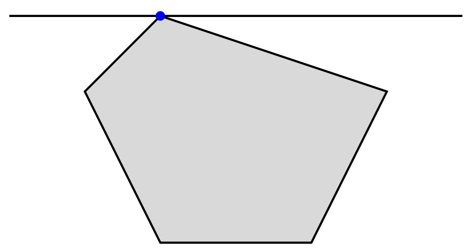

Fig. 5.3: A polyhedron with a valid inequality defining a vertex.

Definition 5.3. A rational polyhedron is called *integral* if each nonempty face of P contains an integer vector.

Lemma 5.1. *Let $P = \{x \in \mathbb{R}^n\colon\ Ax \leqslant b\}$ be an integral polyhedron with $A \in \mathbb{R}^{m\times n}$ full-column rank. If the linear program*

$$\max\{c^T x\colon\ x \in \mathbb{R}^n,\ Ax \leqslant b\} \tag{5.2}$$

is feasible and bounded, then the simplex method computes an optimal integral solution to the linear program.

Proof. Recall that the simplex method finds an optimal roof $B \subseteq \{1,\dots,m\}$ of (5.2) and the vertex of the roof x_B^* is an optimal solution to the linear program (5.2). We have to show that x_B^* is integral. This will follow from the fact that $\{x_B^*\}$ is a face of P.

Proposition 3.1 implies that x_B^* is the unique optimum solution of the linear program $\max\{\tilde{c}^T x\colon\ x \in R^n,\ a_i^T x \leqslant b(i),\ i \in B\}$, where $\tilde{c} = \sum_{i\in B} a_i$. Consequently x_B^* is the unique solution of the linear program

$$\max\{\tilde{c}^T x\colon\ x \in P\}$$

which implies that $\{x_B^*\}$ is a face defined by the valid inequality $\overline{c}^T x \leqslant \overline{c}^T x_B^*$. □

Lemma 5.2. *Let $A \in \mathbb{Z}^{n\times n}$ be an integral and invertible matrix. One has $A^{-1}b \in \mathbb{Z}^n$ for each $b \in \mathbb{Z}^n$ if and only if* $\det(A) = \pm 1$.

Proof. Recall Cramer's rule which says $A^{-1} = 1/\det(A)\widetilde{A}$, where $\widetilde{A}$ is the adjoint matrix of A. Clearly $\widetilde{A}$ is integral. If $\det(A) = \pm 1$, then A^{-1} is an integer matrix.

If $A^{-1}b$ is integral for each $b \in \mathbb{Z}^n$, then A^{-1} is an integer matrix. We have $1 = \det(A \cdot A^{-1}) = \det(A)\cdot\det(A^{-1})$. Since A and A^{-1} are integral it follows that $\det(A)$ and $\det(A^{-1})$ are integers. The only divisors of one in the integers are ± 1. ◻

An integral matrix $A \in \{0,\pm 1\}^{m\times n}$ is called *totally unimodular* if each of its square sub-matrices has determinant $0,\pm 1$.

Theorem 5.2 (Hoffman-Kruskal Theorem). *Let $A \in \mathbb{Z}^{m\times n}$ be an integral matrix. The polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b,\ x \geqslant 0\}$ is integral for each integral $b \in \mathbb{Z}^m$ if and only if A is totally unimodular.*

Proof. Let $A \in \mathbb{Z}^{m\times n}$ be totally unimodular and $b \in \mathbb{Z}^m$. Let x^* be vertex of P and suppose that this vertex is defined by the valid inequality $c^T x \leqslant \delta$. Notice that the matrix $\binom{A}{-I}$ has full column-rank. If one applies the simplex algorithm to the problem

$$\max\{c^T x \colon x \in \mathbb{R}^n,\ \tbinom{A}{-I}x \leqslant \tbinom{b}{0}\},$$

it finds an optimal roof $B \subseteq \{1,\dots,m+n\}$ with $x_B^* = x^*$. If A_B denotes the matrix whose rows are those rows of $\binom{A}{-I}$ indexed by B and if b_B denotes the vector whose components are those of $\binom{b}{0}$ indexed by B, then $x^* = A_B^{-1}b_B$. We are done, once we conclude that $\det(A_B) = \pm 1$, since then A_B^{-1} is an integer matrix and since b_B is an integer vector $x^* = A_B^{-1}b$ is integral as well. We can permute the columns of A_B in such a way that one obtains a matrix of the form

$$\begin{pmatrix} \overline{A} & \widetilde{A} \\ 0 & I_k \end{pmatrix}$$

where $\overline{A}$ is a $(n-k)\times(n-k)$ sub-matrix of A and I_k is the $k\times k$ identity matrix. Here $k = |B \cap \{m+1,\dots,m+n\}$. Clearly $0 \neq \det(A_B) = \pm\det(\overline{A}) = \pm 1$.

For the converse, suppose that A is not totally unimodular. Then there exists an index set $B \subseteq \{1,\dots,m+n\}$ with $|B| = n$ such that the matrix A_B defined as above satisfies $|\det(A_B)| \geqslant 2$. By Lemma 5.2 there exists choices for the components of b_B making $A_B^{-1}b_B$ non-integral. By choosing $\gamma \in \mathbb{N}$ large enough the point $x_B^* = A_B^{-1}(b_B + \gamma A_B \mathbf{1})$ is non-integral and positive. The set B is a non-degenerate roof of the linear program

$$\max\{\overline{c}^T x \colon x \in \mathbb{R}^n,\ \tbinom{A}{-I}x \leqslant \tbinom{b}{0}\},$$

where $\overline{c} = \sum_{i\in B} a_i$ and a_i denotes the i-th row of $\binom{A}{-I}$. If we define for $j \in \{1,\dots,m\}\setminus B$, $b(j) = \lceil a_j^T x_B^* \rceil$, then x_B^* is feasible and thus a vertex of P that is non-integral. ◻

5.2 Applications of total unimodularity

5.2.1 Bipartite matching

A graph is *bipartite*, if V has a partition into sets A and B such that each edge uv satisfies $u \in A$ and $v \in B$. Recall that $\delta(v)$ is the set of edges incident to the vertex $v \in V$, that is $\delta(v) = \{e \in E \mid v \in e\}$.

The *node-edge* incidence matrix of a graph $G = (V, E)$ is the $A \in \{0,1\}^{|V| \times |E|}$ with

$$A(v, e) = \begin{cases} 1, & \text{if } v \in e, \\ 0 & \text{otherwise.} \end{cases}$$

The integer program (5.1) can thus be formulated as

$$\max\{w^T x \colon \; Ax \leqslant 1, \; x \geqslant 0, \; x \in \mathbb{Z}^E\}. \tag{5.3}$$

The next lemma implies that the simplex algorithm can be used to compute a maximum-weight matching of a bipartite graph.

Lemma 5.3. *If G is bipartite, the node-edge incidence matrix of G is totally unimodular.*

Proof (Proof of Lemma 5.3). Let $G = (V, E)$ be a bipartite graph with bi-partition $V = V_1 \cup V_2$.

Let A' be a $k \times k$ sub-matrix of A. We are interested in the determinant of A. Clearly, we can assume that A does not contain a column which contains no 1 or only one 1, since we simply consider the sub-matrix A'' of A', which emerges from developing the determinant of A' along this column. The determinant of A' would be zero or $\pm 1 \cdot \det(A'')$.

Thus we can assume that each column contains exactly two ones. Now we can order the rows of A' such that the first rows correspond to vertices of V_1 and then follow the rows corresponding to vertices in V_2. This re-ordering only affects the sign of the determinant. By summing up the rows of A' in V_1 we obtain exactly the same row-vector as we get by summing up the rows of A' corresponding to V_2. This shows that $\det(A') = 0$. □

5.2.2 Bipartite vertex cover

A *vertex cover* of a graph $G = (V, E)$ is a subset $C \subseteq V$ of the nodes such $e \cap C \neq \emptyset$ for each $e \in E$. Let us formulate an integer program for the *minimum-weight vertex-cover* problem. Here, one is given a graph $G = (V, E)$ and weights $w \in \mathbb{R}^V$. The goal is to find a vertex cover C with minimum weight $w(C) = \sum_{v \in V} w(v)$.

$$\begin{aligned} \min \textstyle\sum_{v\in V} w(v)x(v) \\ uv \in E : x(u) + x(v) \geqslant 1 \\ v \in V : x(v) \geqslant 0 \\ x \in \mathbb{Z}^V. \end{aligned} \tag{5.4}$$

Clearly, this is the integer program

$$\min\{w^T x \colon A^T x \geqslant 1, x \geqslant 0, x \in \mathbb{Z}^V\}, \tag{5.5}$$

where A is the node-edge incidence matrix of G. A matrix A is totally unimodular if and only of A^T is totally unimodular. Thus the simplex algorithm can be used to compute a minimum-weight vertex-cover of a bipartite graph. Furthermore we have the following theorem.

Theorem 5.3 (König's theorem). *In any bipartite graph, the number of edges in a maximum matching equals the number of vertices in a minimum vertex cover.*

Proof. Let A be the node-edge incidence-matrix of the bipartite graph $G = (V,E)$. The linear programs $\max\{1^T x \colon Ax \leqslant 1, x \geqslant 0\}$ and $\min\{1^T x \colon Ax \geqslant 1, x \geqslant 0\}$ are duals of each other. Since A is totally unimodular, the value of the linear programs are the cardinality of a maximum matching and minimum vertex-cover respectively. Thus the theorem follows from strong duality. □

5.2.3 Flows

Let $G = (V,A)$ be a directed graph. The *node-edge incidence matrix of a directed graph* is a matrix $A \in \{0,\pm 1\}^{V\times E}$ with

$$A(v,a) = \begin{cases} 1 & \text{if } v \text{ is the starting-node of } a, \\ -1 & \text{if } v \text{ is the end-node of } a, \\ 0 & \text{otherwise.} \end{cases} \tag{5.6}$$

A *feasible flow* f of G with capacities u and in-out-flow b is then a solution $f \in \mathbb{R}^A$ to the system $Af = b$, $0 \leqslant f \leqslant u$.

Lemma 5.4. *The node-edge incidence matrix A of a directed graph is totally unimodular.*

Proof. Let A' be a $k \times k$ sub-matrix of A. Again, we can assume that in each column we have exactly one 1 and one -1. Otherwise, we develop the determinant along a column which does not have this property. But then, the A' is singular, since adding up all rows of A' yields the 0-vector.

A consequence is that, if the b-vector and the capacities u are integral and an optimal flow exists, then there exists an integer optimal flow.

5.2.4 *Doubly stochastic matrices*

A matrix $A \in \mathbb{R}^{n\times n}$ is *doubly stochastic* if it satisfies the following linear constraints

$$\begin{aligned} \sum_{i=1}^n A(i,j) &= 1, \; \forall j = 1,\dots,n \\ \sum_{j=1}^n A(i,j) &= 1, \; \forall i = 1,\dots,n \\ A(i,j) &\geqslant 0, \; \forall 1 \leqslant i,j \leqslant n. \end{aligned} \tag{5.7}$$

A permutation matrix is a matrix which contains exactly one 1 per row and column, where the other entries are all 0.

Theorem 5.4. *A matrix $A \in \mathbb{R}^{n\times n}$ is doubly stochastic if and only if A is a convex combination of permutation matrices.*

Proof. Since a permutation matrix satisfies the constraints (5.7), then so does a convex combination of these constraints.

On the other hand it is enough to show that each vertex of the polytope defined by the system (5.7) is integral and thus a permutation matrix. However, the matrix defining the system (5.7) is the node-edge incidence matrix of the complete bipartite graph having $2n$ vertices. Since such a matrix is totally unimodular, the theorem follows.

5.3 The matching polytope

We now come to a deeper theorem concerning the convex hull of matchings. We mentioned several times in the course that the maximum weight matching problem can be solved in polynomial time. We are now going to show a theorem of Edmonds [1] which provides a complete description of the matching polytope and present the proof by Lovász [10].

Before we proceed let us inspect the symmetric difference $M_1 \Delta M_2$ of two matchings of a graph G. If a vertex is adjacent to two edges of $M_1 \cup M_2$, then one of the two edges belongs to M_1 and one belongs to M_2. Also, a vertex can never be adjacent to three edges in $M_1 \cup M_2$. Edges which are both in M_1 and M_2 do not appear in the symmetric difference. We therefore have the following lemma.

Lemma 5.5. *The symmetric difference $M_1 \Delta M_2$ of two matchings decomposes into node-disjoint paths and cycles, where the edges on these paths and cycles alternate between M_1 and M_2.*

The *Matching polytope* $P(G)$ of an undirected graph $G = (V,E)$ is the convex hull of incidence vectors χ^M of matchings M of G.

The incidence vectors of matchings are exactly the $0/1$-vectors that satisfy the following system of equations.

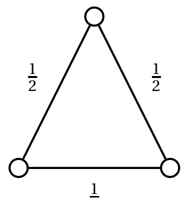

Fig. 5.4: Triangle

$$\begin{aligned} \textstyle\sum_{e\in\delta(v)} x(e) &\leqslant 1 \;\; \forall v \in V \\ x(e) &\geqslant 0 \;\; \forall e \in E. \end{aligned} \tag{5.8}$$

However the triangle (Figure 5.4) shows that the corresponding polytope is not integral. The objective function $\max 1^T x$ has value 1.5. However, one can show that a maximum weight matching of an undirected graph can be computed in polynomial time which is a result of Edmonds [2].

The following (Figure 5.5) is an illustration of an Edmonds inequality. Suppose that U is an odd subset of the nodes V of G and let M be a matching of G. The number of edges of M with both endpoints in U is bounded from above by $\lfloor |U|/2 \rfloor$.

Thus the following inequality is valid for the integer points of the polyhedron defined by (5.8).

$$\sum_{e\in E(U)} x(e) \leqslant \lfloor |U|/2 \rfloor, \qquad \text{for each } U \subseteq V, \quad |U| \equiv 1 \pmod 2. \tag{5.9}$$

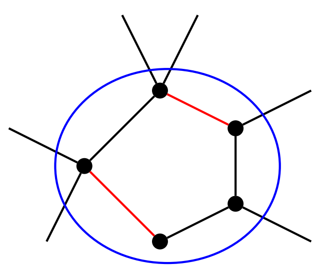

Fig. 5.5: Edmonds inequality.

The goal of this lecture is a proof of the following theorem.

Theorem 5.5 (Edmonds 65). *The matching polytope is described by the following inequalities:*

i) $x(e) \geqslant 0$ *for each* $e \in E$,
ii) $\sum_{e\in\delta(v)} x(e) \leqslant 1$ *for each* $v \in V$,
iii) $\sum_{e\in E(U)} x(e) \leqslant \lfloor |U|/2 \rfloor$ *for each* $U \subseteq V$

Lemma 5.6. *Let* $G = (V, E)$ *be connected and let* $w : E \longrightarrow \mathbb{R}_{>0}$ *be a weight-function. Denote the set of maximum weight matchings of G w.r.t. w by* $\mathscr{M}(w)$. *Then one of the following statements must be true:*

i) $\exists v \in V$ *such that* $\delta(v) \cap M \neq \emptyset$ *for each* $M \in \mathscr{M}(w)$
ii) $|M| = \lfloor |V|/2 \rfloor$ *for each* $M \in \mathscr{M}(w)$ *and* $|V|$ *is odd.*

Proof. Suppose both $i)$ and $ii)$ do not hold. Then there exists $M \in \mathscr{M}(w)$ leaving two exposed nodes u and v. Choose M such that the minimum distance between two exposed nodes u, v is minimized.

Now let t be on shortest path from u to v. The vertex t cannot be exposed.

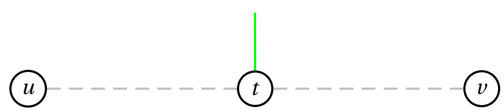

Fig. 5.6: Shortest path between u and v.

Let $M' \in \mathscr{M}(w)$ leave t exposed. Both u and v are covered by M' because the distance to u or v from t is smaller than the distance of u to v.

Consider the symmetric difference $M \triangle M'$ which decomposes into node disjoint paths and cycles. The nodes u, v and t have degree one in $M \triangle M'$. Let P be a path with endpoint t in $M \triangle M'$

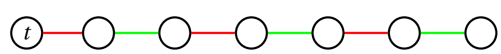

Fig. 5.7: Swapping colors.

If we swap colors on P, see Figure 5.7, we obtain matchings $\widetilde{M}$ and $\widetilde{M'}$ with $w(M) + w(M') = w(\widetilde{M}) + w(\widetilde{M'})$ and thus $\widetilde{M} \in \mathscr{M}(w)$.

The node t is exposed in $\widetilde{M}$ and u or v is exposed in $\widetilde{M}$. This is a contradiction to u and v being shortest distance exposed vertices

Proof (Proof of Theorem 5.5).

Let $w^T x \leqslant \beta$ be a *facet* of $P(G)$, we need to show that this facet it is of the form

i) $x(e) \geqslant 0$ for some $e \in E$

ii) $\sum_{e\in\delta(v)} x(e) \leqslant 1$ for some $v \in V$
iii) $\sum_{e\in E(U)} x(e) \leqslant \lfloor |U|/2 \rfloor$ for some $U \in P_{odd}$

To do so, we use the following method: One of the inequalities i), ii), iii) is satisfied with equality by each χ^M, $M \in \mathscr{M}(w)$. This establishes the claim since the matching polytope is full-dimensional and a facet is a maximal face.

If $w(e) < 0$ for some $e \in E$, then each $M \in \mathscr{M}(w)$ satisfies $e \notin M$ and thus satisfies $x(e) \geqslant 0$ with equality.

Thus we can assume that $w \geqslant 0$.

Let $G^* = (V^*, E^*)$ be the graph induced by edges e with $w(e) > 0$. Each $M \in \mathscr{M}(w)$ contains maximum weight matching $M^* = M \cap E^*$ of G^* w.r.t. w^*.

If G^* is not *connected* , suppose that $V^* = V_1 \cup V_2$, where $V_1 \cap V_2 = \emptyset$ and $V_1, V_2 \neq \emptyset$ and there is no edge connecting V_1 and V_2, then $w^T x \leqslant \beta$ can be written as the sum of $w_1^T x \leqslant \beta_1$ and $w_2^T x \leqslant \beta_2$, where β_i is the maximum weight of a matching in V_i w.r.t. w_i, $i = 1,2$, see Figure 5.8. This would also contradict the fact that $w^T x \leqslant \beta$ is a facet, since it would follow from the previous inequalities and thus would be a redundant inequality.

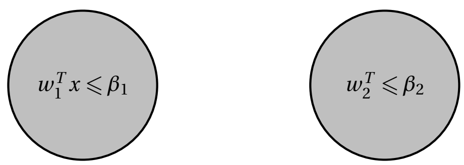

Fig. 5.8: G^* is connected.

Now we can use Lemma 5.6 for G^*.

i) $\exists v$ such that $\delta(v) \cap M = \emptyset$ for each $M \in \mathscr{M}(w)$. This means that each M in $\mathscr{M}(w)$ satisfies

$$\sum_{e\in\delta(v)} x(e) \leqslant 1 \quad \text{with equality}$$

ii) $|M \cap E^*| = \lfloor |V^*|/2 \rfloor$ for each $M \in \mathscr{M}(w)$ and $|V^*|$ is odd. This means that each M in $\mathscr{M}(w)$ satisfies

$$\sum_{e\in E(V^*)} x(e) \leqslant \lfloor |V^*|/2 \rfloor \quad \text{with equality}$$

Exercises

1. Let $M \in \mathbb{Z}^{n\times m}$ be totally unimodular. Prove that the following matrices are totally unimodular as well:

   i) M^T

ii) $(M \quad I_n)$
iii) $(M \quad -M)$
iv) $M \cdot (I_n - 2e_j e_j^T)$ for some j

I_n is the $n \times n$ identity matrix, and e_j is the vector having a 1 in the jth component, and 0 in the other components.

2. A family $\mathscr{F}$ of subsets of a finite groundset E is *laminar*, if for all $C, D \in \mathscr{F}$, one of the following holds:

$$(i)\ C \cap D = \emptyset, \quad (ii)\ C \subseteq D, \quad (iii)\ D \subseteq C.$$

Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two laminar families of the same groundset E and consider its union $\mathscr{F}_1 \cup \mathscr{F}_2$. Define the $|\mathscr{F}_1 \cup \mathscr{F}_2| \times |E|$ adjacency matrix A as follows: For $F \in \mathscr{F}_1 \cup \mathscr{F}_2$ and $e \in E$ we have $A_{F,e} = 1$, if $e \in F$ and $A_{F,e} = 0$ otherwise. Show that A is totally unimodular.

3. Consider the following scheduling problem: Given n tasks with periods $p_1, \ldots, p_n \in \mathbb{N}$, we want to find offsets $x_i \in \mathbb{N}_0$, such that every task i can be executed periodically at times $x_i + p_i \cdot k$ for all $k \in \mathbb{N}_0$. In other words, for all pairs i, j of tasks we require $x_i + k \cdot p_i \neq x_j + l \cdot p_j$ for all $k, l \in \mathbb{N}_0$.
Formulate the problem of finding these offsets as an integer program (with zero objective function).

4. Let $P = \{x \in \mathbb{R}^n \colon Ax \leqslant b\}$ be a polyhedron. Show that the following are equivalent for a feasible x^*:

   i) x^* is a vertex of P.
   ii) There exists a set $B \subseteq \{1, \ldots, m\}$ such that $|B| = n$, A_B is invertible and $A_B x^* = b_B$. Here the matrix A_B and the vector b_B consists of the rows of A indexed by B and the components of b indexed by B respectively.
   iii) For every feasible $x_1, x_2 \neq x^* \in P$ one has $x^* \notin \operatorname{conv}\{x_1, x_2\}$.

5. Show the following: A polyhedron $P \subseteq \mathbb{R}^n$ with vertices is integral, if and only if each vertex is integral.

6. Consider the polyhedron $P = \{x \in \mathbb{R}^3 \colon x_1 + 2x_2 + 4x_3 \leqslant 4,\ x \geqslant 0\}$. Show that this polyhedron is integral.

7. Which of these matrices is totally unimodular? Justify your answer.

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

8. Consider the complete graph G_n with 3 vertices, i.e., $G = (\{1,2,3\}, \binom{3}{2})$. Is the polyhedron of the linear programming relaxation of the vertex-cover integer program integral?

References

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