# Chapter 3
# The simplex method

In this chapter, we describe the simplex method. The task is to solve a linear program

$$\max\{c^T x\colon x \in \mathbb{R}^n,\, Ax \leqslant b\} \tag{3.1}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. We make the following assumption.

| FULL-RANK ASSUMPTION |
|---|
| The matrix $A \in \mathbb{R}^{m\times n}$ has full column-rank. In other words, the columns of $A$ are linearly independent |

We will see later that this assumption can be made without loss of generality.

## 3.1 Roofs

Roofs are linear programs originating from (3.1) by selecting a subset of the inequalities only. A roof should provide an upper bound on the optimal value of the linear program (3.1) and at the same time consist of $n$ "linearly independent constraints". Here is the definition of a roof.

**Definition 3.1.** Consider the linear program (3.1) and let $B \subseteq \{1,\dots,m\}$ be a subset of the row-indices. This set $B$ is a *roof* if

i) $|B| = n$,
ii) The rows $a_i$, $i \in B$ are linearly independent, and
iii) The linear program

$$\max\{c^T x\colon a_i^T x \leqslant b(i),\, i \in B\} \tag{3.2}$$

is bounded.

What is the optimal solution of a linear program (3.2) defined by a roof? This question is answered in the next lemma.

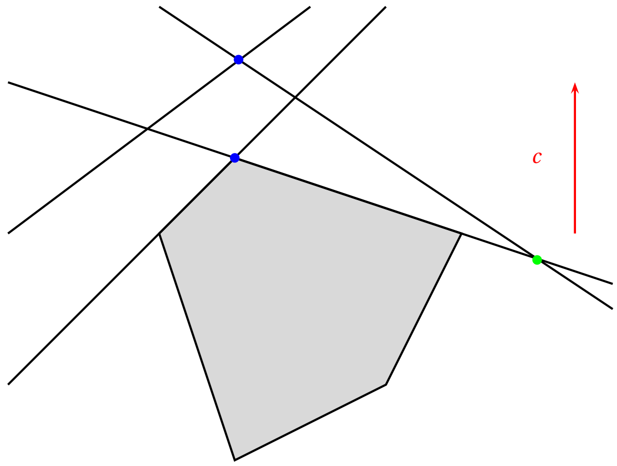


**Fig. 3.1** A linear program; the objective function vector $c$ is pointing vertically upwards. The blue dots mark two roofs. Notice that the lowest roof is the optimum of the linear program. The green point marks a non-roof. The two constraints satisfy i) and ii) but not iii).

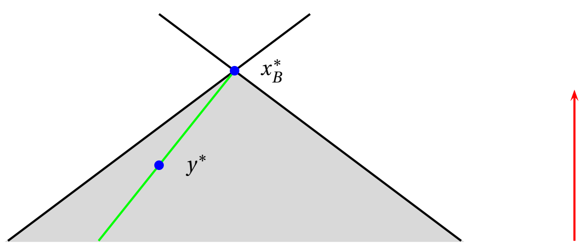


**Fig. 3.2** An illustration for the proof of Lemma 3.1. The green ray illustrates the set $\{x_B^* + \lambda(y^* - x_B^*) \colon \lambda \in \mathbb{R}_{\geqslant 0}\}$.

**Lemma 3.1.** *Let $B \subseteq \{1,\dots,m\}$ be a roof of the linear program* (3.1) *and let $x_B^*$ be the unique solution of the linear system*

$$a_i^T x = b(i),\ i \in B,$$

*then $x_B^*$ is an optimal solution of the roof-linear program* (3.2).

*Proof.* Suppose that $y^*$ is a feasible solution of the roof-linear program with $c^T y^* > c^T x_B^*$. We now show that each $x_B^* + \lambda(y^* - x_B^*)$ for $\lambda \geqslant 0$ is feasible. One has $a_i^T(x_B^* + \lambda(y^* - x_B^*)) = b(i) + \lambda \cdot (a_i^T y^* - b(i)) \leqslant b(i)$ for each $i \in B$ and thus $x_B^* + \lambda(y^* - x_B^*)$ is feasible for each $\lambda \geqslant 0$.

The objective function value of such a point is $c^T x_B^* + \lambda(c^T y^* - c^T x_B^*)$ which, for $\lambda \to \infty$ tends to infinity. This is a contradiction to $B$ being a roof (condition iii). Thus $x_B^*$ must be an optimal solution to the roof-linear program (3.2). □

Now that we know that a roof-linear program has an optimal solution, we can define the value of a roof $B$.

**Definition 3.2.** The *value* of a roof $B$ is the optimum value $c^T x_B^*$ of the roof-linear program

$$\max\{c^T x \colon a_i^T x \leqslant b(i),\, i \in B\}.$$

The next theorem is very simple, but in fact very important. It states that the value of a roof is an upper bound on the optimum value of a linear program $\max\{c^T x \colon x \in \mathbb{R}^n,\, Ax \leqslant b\}$.

**Theorem 3.1 (Weak duality).** *The value of a roof is an upper bound on the objective function value of any feasible point of the linear program.*

*Proof.* Let $B$ be a roof of the linear program $\max\{c^T x \colon x \in \mathbb{R}^n,\, Ax \leqslant b\}$. Any feasible point $x^*$ of this linear program is also a feasible point of the roof-linear program $\max\{c^T x \colon a_i^T x \leqslant b(i)\}$. Therefore $c^T x_B^* \geqslant c^T x^*$ and the claim follows. □

When is an index-set $B \subseteq \{1, \ldots, m\}$ satisfying i) and ii) a roof? Consider the example in see Figure 6.6. The objective is to maximize $2x_1 + x_2$ and the two roof-constraints are $x_1 + x_2 \leqslant 5$ and $x_1 \leqslant 6$. From the picture, it is clear that the objective function vector is in the cone of the two constraint vectors. In fact, this is the characterization that holds in any dimension as we now show.

**Lemma 3.2.** *Let $B \subseteq \{1, \ldots, m\}$ satisfy i) and ii). Then $B$ is a roof, if and only if $c \in$* $\operatorname{cone}\{a_i \colon i \in B\}$.

*Proof.* Suppose that $c \in \operatorname{cone}\{a_i \colon i \in B\}$. Thus there exist $\lambda_i \geqslant 0$, $i \in B$ with $c = \sum_{i \in B} \lambda_i \cdot a_i$. The unique solution $x_B^*$ to the system

$$a_i^T x = b(i),\, i \in B \tag{3.3}$$

is an optimal solution to $\max\{c^T x \colon a_i^T x \leqslant b(i)\}$. Because if $\widetilde{x}$ is another feasible solution, then $c^T \widetilde{x} = \sum_{i \in B} \lambda_i \cdot a_i^T \widetilde{x}$. Since $\lambda \geqslant 0$ and $a_i^T \widetilde{x} \leqslant b(i)$ we can write

$$\begin{aligned}
c^T \widetilde{x} &= \sum_{i \in B} \lambda_i \cdot a_i^T \widetilde{x} && (3.4)\\
&\leqslant \sum_{i \in B} \lambda_i \cdot b(i) && (3.5)\\
&= \sum_{i \in B} \lambda_i \cdot a_i^T x_B^* && (3.6)\\
&= (\sum_{i \in B} \lambda_i \cdot a_i)^T x_B^* && (3.7)\\
&= c^T x_B^*. && (3.8)
\end{aligned}$$

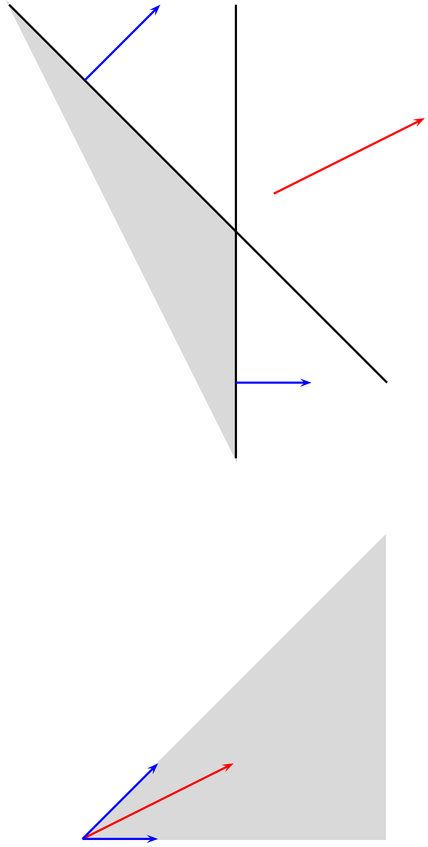

**Fig. 3.3** A roof that is defined by the two constraints $x_1 \leqslant 2$ and $x_1 + x_2 \leqslant 3$. The objective function vector is $(2,1)^T$. Indeed $(1,1)^T = 1\cdot(1,1)+1\cdot(1,0)$ which shows that it is in the cone generated by the constraint-defining vectors.

Thus $B$ is a roof.

Suppose on the other hand that $B$ is a roof. Then, since $a_i, i \in B$ is a basis of $\mathbb{R}^n$, there exist $y_i \in \mathbb{R}$, $i \in B$ with $c = \sum_{i\in B} y_i \cdot a_i$. If all $y_i \geqslant 0$, $i \in B$, then it follows that $c \in \text{cone}\{a_i \colon i \in B\}$ and we are done. Suppose therefore that there exists an index $i' \in B$ with $y_{i'} < 0$. We will derive a contradiction.

Consider the system of linear equations

$$a_{i'}^T x = -1, a_i^T x = 0,\ i \in B \setminus \{i'\}. \tag{3.9}$$

This system (3.9) has a unique solution $0 \neq v \in \mathbb{R}^n$. Let $x^*$ be a feasible solution to the roof-linear program. Clearly $x^* + \lambda \cdot v$ is also feasible for each $\lambda > 0$ (Exercise 2). But $c^T(x^* + \lambda v) = c^T x^* + \lambda \cdot \sum_{i=1}^n y(i) a_i^T v = c^T x^* + \lambda \cdot y_{i'} \cdot a_{i'}^T v$. This increases with

$\lambda$ since $y_{i'} < 0$ and $a_{i'}^T v < 0$. This contradicts the fact $B$ is a roof, since the roof-linear program is unbounded. ◻

**Definition 3.3.** Let $B$ be a roof of the linear program (3.1). The unique solution $x^*$ of the system

$$a_i^T x = b(i),\ i \in B, \tag{3.10}$$

is the *vertex* of the roof.

Similarly one can prove the following fact.

**Proposition 3.1.** *Let $B$ be a roof of the linear program* (3.1). *The vertex of a roof is the* unique optimal solution *of the roof-linear program* (3.2) *if and only if $c$ is a* strictly positive *conic combination of the normal-vectors $a_i$, $i \in B$.*

## 3.2 The simplex algorithm

We now sketch one iteration of the simplex algorithm. Our task is to solve a linear program (3.1) and we assume that we have a roof $B$ to start with.

i) Compute the vertex $x_B^*$ of the roof $B$.
ii) Find an index $i \in \{1,\dots,m\} \setminus B$ with $a_i^T x_B^* > b(i)$. If there does not exist such an index, then $x_B^*$ is an optimal solution of the linear program (3.1).
iii) Determine an index $j \in B$ such that

   a) $B' = B \cup \{i\} \setminus \{j\}$ is a roof, and
   b) The vertex $x_{B'}^*$ of $B'$ is feasible for $B$.

   If such an index does not exist, then the linear program (3.1) is infeasible.

The simplex algorithm iterates these steps until it has found an optimal solution, or asserts that the linear program (3.1) is infeasible. The big questions are how to determine an index $j$ such that iiia) and iiib) hold in step iii) and that the algorithm is correct. Furthermore, we want to understand whether the simplex method eventually terminates.

### *3.2.1 Termination and degeneracy*

**Definition 3.4 (Degenerate roof and linear program).** A roof $B$ of a linear program (3.1) is *degenerate* if the optimum solution of the roof-linear program (3.2) is not unique. A linear program is called degenerate, if it has degenerate roofs.

We now argue that the simplex algorithm terminates if the linear program is non-degenerate.

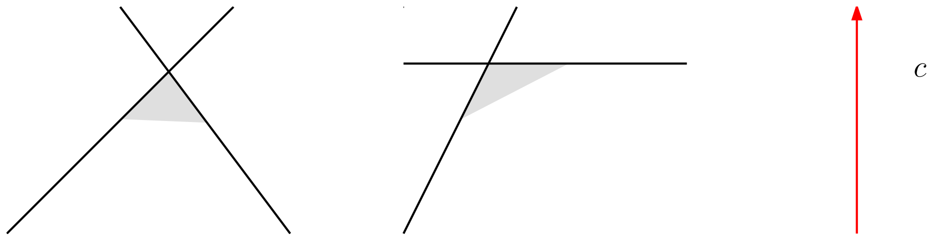


**Fig. 3.4** A non-degenerate and a degenerate roof.

**Theorem 3.2.** *If the linear program* (3.1) *is non-degenerate, then the simplex algorithm terminates.*

*Proof.* The important observation is that the simplex method makes progress from iteration to iteration because of the non-degeneracy of the roofs. If $B'$ is the roof computed in step iii), then, since $x^*_{B'}$ is contained in the feasible region of the roof $B$, and since $B$ is non-degenerate, we have $c^T x^*_B > c^T x^*_{B'}$. Since there is only a finite number of roofs, the algorithm thus terminates. □

### *3.2.2 Implementing step iii)*

The situation is as follows. We are having a roof $B$ and its vertex $x^*_B$ and an index $i \in \{1,\ldots,m\}$ with $a_i^T x^*_B > b(i)$. We now want to bring $i$ into the new roof and we have to determine a $j \in B$ that is supposed to leave the roof. The idea is very similar now to the proof of Carathéodory's theorem.

Consider the systems of equations

$$\sum_{k\in B} a_k x_k + a_i x_i = c \tag{3.11}$$

$$\sum_{k\in B} a_k y_k + a_i y_i = 0 \tag{3.12}$$

with variables $x_k$, $k \in B$, $x_i$ and $y_k$, $k \in B$, $y_i$.

Compute a solution $x^* \in \mathbb{R}^{n+1}$ of (3.11) with component $x^*_i = 0$ and compute solution $y^* \in \mathbb{R}^{n+1}$ of (3.12) with component $y^*_i = 1$. The solutions to (3.11) are the points on the line $\{x^* + \lambda y^* \mid \lambda \in \mathbb{R}\}$. Notice that $x^* \geqslant 0$.

To bring the index $i$ into the roof, we want to increase $\lambda = 0$ until some other component of $x^* + \lambda y^*$, component $j$ lets say, becomes zero. So in virtue of finding an index which drops out of $B$, we have to determine the largest $\lambda^* \in \mathbb{R}_{\geqslant 0}$ such that all components of $x^* + \lambda y^*$ are nonnegative. This is done as follows.

Compute the index set $J = \{k \in B \colon y^*_k < 0\}$. Those are the indices we have to worry about, since only those components can become negative with increasing $\lambda$. Still, how large can $\lambda^*$ be? We have to ensure that

$$x^*(k) + \lambda^* y^*(k) \geqslant 0 \text{ for all } k \in J. \tag{3.13}$$

In other words we have to ensure

$$\lambda^* \leqslant -\frac{x^*(k)}{y^*(k)} \text{ for all } k \in J. \tag{3.14}$$

If $J \neq \emptyset$, we pick

$$\lambda^* = \min_{k \in J} -\frac{x^*(k)}{y^*(k)}. \tag{3.15}$$

We choose an index $j \in J$ for which this minimum is achieved. This index $j$ is the one which leaves the roof.

**Lemma 3.3.** *The index set $B' = B \backslash \{i\} \cup \{j\}$ is a roof and the new vertex $x^*_{B'}$ is contained in the feasible set of the roof $B$.*

*Proof.* By construction, $c$ is a nonnegative linear combination of the vectors $a_k$, $k \in B'$. Thus in order to conclude that $B'$ is a roof, we need to show that the $a_k$, $k \in B'$ are linearly independent. The component $y^*_j$ is nonzero. Since $y^*$ is a solution of equation (3.12) it follows that $a_j$ is a linear combination of the normal-vectors of $B'$. Thus the $a_k$, $k \in B'$ are a basis of $\mathbb{R}^n$ and since $|B'| = n$ they are linearly independent.

Let $x^*_B$ be the vertex of $B$ and let $w \in \mathbb{R}^n$ be a solution to the system

$$a_j^T w = -1, a_k^T w = 0, \, k \in B \setminus \{j\}. \tag{3.16}$$

The half-line $l(x^*_B, w) = \{x^* + \lambda\, w \mid \lambda \in \mathbb{R}_{\geqslant 0}\}$ is feasible for $B$. We have the equation

$$a_i = -\sum_{k \in B} y^*_k a_k \tag{3.17}$$

where $y^*_j < 0$. Thus

$$a_i^T w = -\sum_{k \in B} y^*_k a_k^T w \tag{3.18}$$

$$= y^*_k \tag{3.19}$$

$$< 0. \tag{3.20}$$

Therefore the hal-fline $l(x^*, w)$ enters at some point $x'$ the halfspace $a_i^T x \leqslant b(i)$. This is the vertex $x^*_{B'}$ of $B'$. □

*Example 3.1.* Consider the linear program $\max\{x_2 \colon x \in \mathbb{R}^n, (-1,1)x \leqslant 1, (2,1)x \leqslant 1, (1,2)x \leqslant 1\}$. We start with the roof $B = \{1,2\}$ that consists of the first two inequalities see Figure 3.5.

We compute first the vertex $x^*_B$ which is the solution to the system

$$\begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{3.21}$$

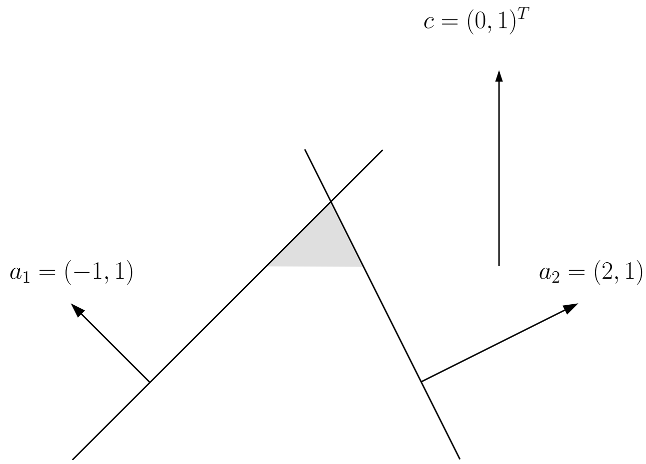


**Fig. 3.5** The initial roof of Example 3.1.

Thus the vertex is the vector $x_B^* = \binom{0}{1}$.
Next we find that the halfspace $(1,2)x \leqslant 1$ is not satisfied by $x_B^* = \binom{0}{1}$. We want to bring this index into the new roof $B'$.
Step 3: Now we compute the solution to the system

$$\begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} x = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.22}$$

and find

$$x^* = \begin{pmatrix} 2/3 \\ 1/3 \\ 0 \end{pmatrix}.$$

Next we find a solution to the system

$$\begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y(1) \\ y(2) \end{pmatrix} = - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{3.23}$$

and find

$$y^* = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

The index set $J = \{1,2\}$ is not empty. The minimum (3.15) is achieved at $j = 2$. So the halfspace $(2,1)x \leqslant 1$ will leave the roof and $B' = \{1,3\}$. This is also what we immediately see by looking at Figure 3.6.

If $J = \emptyset$, we assert that the linear program is infeasible based on the following result.

**Proposition 3.2.** *The half-spaces $a_k^T x \leqslant b(k)$, $k \in B$ and $a_i^T x \leqslant b(i)$ define together an infeasible system if and only if $J = \emptyset$.*

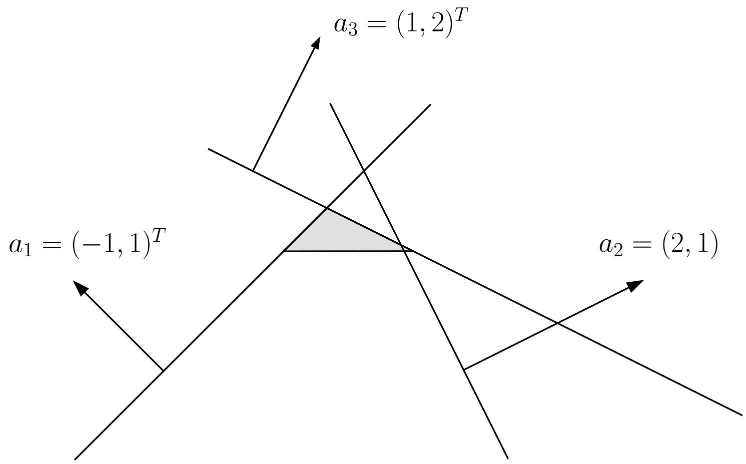


**Fig. 3.6** The new roof from Example 3.1.

*Proof.* The index set $J$ is empty if and only if $y^* \geqslant 0$. Let $\tilde{x}$ be feasible for $B$. We now show that $\tilde{x}$ is not feasible for $a_i^T x \leqslant b(i)$. Remember, that $x_B^*$ does not satisfy $a_i^T x \leqslant b(i)$.

$$\begin{aligned} -a_i^T \tilde{x} &= \sum_{k \in B} y_k^* a_k^T \tilde{x} \\ &\leqslant \sum_{k \in B} y_k^* b(k) \\ &= \sum_{k \in B} y_k^* a_k^T x_B^* \\ &= -a_i^T x_B^* \\ &< -b(i). \end{aligned}$$

Similarly we can see that, if a point $\tilde{x}$ is feasible for all halfspaces in $B$ and the halfspace $a_i^T x \leqslant b(i)$, then $y^*$ cannot be non-negative. Because we proved above that $y^* \geqslant 0$ implies that each feasible point for $B$ is infeasible for $a_i^T x \leqslant b(i)$. □

### 3.2.3 *The degenerate case*

The termination argument for the non-degenerate case was that the value of the new roof is strictly dropping and thus, that a roof can never be revisited. Since there are only a finite number of roofs, this implies that the simplex algorithm terminates.

In the degenerate case, roofs could be revisited. This phenomenon is called *cycling* and you are asked to construct such an example in the exercises. What can we do about it? The idea is to change the objective vector $c \in \mathbb{R}^n$ a little bit and turn it into a vector $c_\epsilon$ such that the following conditions hold.

1) The linear program
$$\max\{c_\epsilon^T x\colon x \in \mathbb{R}^n,\, Ax \leqslant b\} \tag{3.24}$$
has a roof.
2) Each non-roof of the linear program (3.1) is a non-roof of the linear program (3.24).
3) No roof of (3.24) is degenerate.

Suppose we have an initial roof $B = \{i_1, \dots, i_n\}$ at the beginning of the simplex algorithm and let $A_B \in \mathbb{R}^{n\times n}$ be the matrix whose rows are the vectors $a_{i_j}$, $j = 1, \dots, n$. The system ${A_B}^T y = c$ has a solution $y^* \geqslant 0$, where some components of $y^*$ are zero if and only if $B$ is degenerate. This is undesirable and we wish that $y^*$ is replaced by
$$y^* + \begin{pmatrix} \epsilon \\ \epsilon^2 \\ \vdots \\ \epsilon^n \end{pmatrix} \tag{3.25}$$
for some $\epsilon > 0$. Later it will become clear why we add the vector $(\epsilon, \dots, \epsilon^n)^T$ instead of the vector $(\epsilon, \dots, \epsilon)^T$. Now the vector (3.25) becomes a solution if we perturb $c$ and consider the vector
$$c_\epsilon = c + {A_B}^T \begin{pmatrix} \epsilon \\ \epsilon^2 \\ \vdots \\ \epsilon^n \end{pmatrix} \tag{3.26}$$
instead. If $\epsilon > 0$, then $B$ is a non-degenerate roof of the linear program (3.24). Thus condition 1) holds for any $\epsilon > 0$. In the sequel, we make $\epsilon$ smaller and smaller, such that also the conditions 2) and 3) will be satisfied.

Let us first deal with condition 2). Let $B'$ be a set of linear indices such that the vectors $a_i$, $i \in B'$ are a basis of $\mathbb{R}^n$ and suppose that $B'$ is not a roof. We have to guarantee that $B'$ is not a roof of the perturbed linear program.

Let $A_{B'} \in \mathbb{R}^{n\times n}$ be the sub-matrix of $A$ that is defined by the rows of $A$ indexed by $B'$. Since $B'$ is not a roof, the vector ${A_{B'}}^{-T} c$ has a strictly negative component. Suppose that this component is the $i$-th component $(A_{B'}^{-T} c)(i) < 0$. By choosing $\epsilon > 0$ sufficiently small, we guarantee that

$$(A_{B'}^{-T} c_\epsilon)(i) = (A_{B'}^{-T} c + A_{B'}^{-T} A_B^T \begin{pmatrix} \epsilon \\ \epsilon^2 \\ \vdots \\ \epsilon^n \end{pmatrix})(i) \tag{3.27}$$

$$< 0. \tag{3.28}$$

Thus condition 2) is satisfied by choosing $\epsilon > 0$ sufficiently small.

For condition 3) we have to work only a little harder, and in fact, this is why we add the vector $(\epsilon, \dots, \epsilon^n)$ instead of $(\epsilon, \dots, \epsilon)$. Let $B'$ be a roof of (3.24) and let $A_{B'} \in \mathbb{R}^{n \times n}$ be again the above described sub-matrix. We now argue that, if $\epsilon$ is sufficiently small, then $A_{B'}^{-T} c_\epsilon$ does not have any component equal to zero. We are done then in showing that, for $\epsilon$ sufficiently small, the linear program (3.24) is non-degenerate. Because any roof of (3.24) will then be non-degenerate.

So let us inspect the vector

$$A_{B'}^{-T} c_\epsilon = A_{B'}^{-T} c + A_{B'}^{-T} \cdot A_B{}^T \begin{pmatrix} \epsilon \\ \vdots \\ \epsilon^n \end{pmatrix}. \tag{3.29}$$

Each component of (3.29) is a nonzero polynomial with variable $\epsilon$. Nonzero, because $A_{B'}$ and $A_B$ are non-singular matrices. It is well known, that a nonzero polynomial of degree $n$ has at most $n$ roots. Thus, for $\epsilon > 0$, sufficiently small, no component of (3.29) will be zero.

So the conditions 1), 2) and 3) hold for $\epsilon > 0$ sufficiently small. Thus we can modify a degenerate linear program into an equivalent non-degenerate linear program and apply the simplex algorithm to it.

**Theorem 3.3.** *Let*

$$\max\{c^T x \colon x \in \mathbb{R}^n,\, Ax \leqslant b\} \tag{3.30}$$

*be a linear program and let $B$ be a roof. The simplex method terminates on the perturbed linear program* (3.24)*. It either returns a roof $B'$ of* (3.30) *and* (3.24) *whose vertex $x_{B'}^*$ is an optimal solution of* (3.30) *or it asserts that* (3.30) *is infeasible.*

*Proof.* The simplex method terminates on (3.24) since this linear program is non-degenerate. If it asserts that (3.26) is infeasible, then it also follows that (3.30) is infeasible, since the perturbation only changes the objective-function vector.

If it returns a roof $B'$ of (3.24), then this is also a roof of (3.30) by condition 2). Furthermore, the vertex $x_{B'}^*$ is feasible for (3.30). It follows from weak duality (Theorem 3.1) that $x_{B'}^*$ is an optimal solution of (3.30). □

## 3.3 Phase I, finding an initial roof

So far, we always started with an initial roof. Where do we get it from? This is where Phase I of the simplex method is put to work. Above, we described Phase II. First we prove a little lemma.

**Lemma 3.4.** *If the linear program* (3.1) *is feasible and bounded, then it has an optimal roof. In particular, a feasible and bounded linear program has an optimal solution.*

*Proof.* We change the linear program (3.1) by adding the additional constraint $c^T x \leqslant M + 1$, where $c^T x \leqslant M$ is valid for all feasible solutions. We then have a (degenerate) roof by choosing this inequality together with any subset of $n-1$ constraints whose normal-vectors together with $c$ form a basis of $\mathbb{R}^n$. The simplex algorithm will find an optimal roof. $\square$

We now form an auxiliary linear program. Observe that the linear program

$$\max\{c^T x \colon x \in \mathbb{R}^n,\ Ax \leqslant b\} \tag{3.31}$$

has a roof if and only if the linear program $\max\{c^T x \colon x \in \mathbb{R}^n,\ Ax \leqslant 0\}$ has a roof. The latter linear program is feasible since 0 is a feasible solution. Furthermore, 0 is an optimal solution of this linear program if and only if it has a roof. This is what we check with the simplex algorithm on the auxiliary program

$$\max\{c^T x \colon x \in \mathbb{R}^n,\ Ax \leqslant 0,\ c^T x \leqslant 1\} \tag{3.32}$$

and we start with a (possibly) degenerate roof involving the inequality $c^T x \leqslant 1$ and $n-1$ of the constraints of $Ax \leqslant 0$ whose normal-vectors, together with $c$ form a basis of $\mathbb{R}^n$. The simplex algorithm terminates with an optimal roof. If the roof has vertex 0, then we have found a roof of the original linear program (3.31) and can start the simplex algorithm for it. If the roof of Phase I still contains the inequality $c^T x \leqslant 1$, then the original linear program (3.31) does not have any roof. This either means that the program is infeasible or unbounded.

## 3.4 The full column-rank assumption

There is one thing that we still have to deal with. The simplex algorithm is based on the assumption that the columns of $A$ are linearly independent. We now argue that this assumption can be made without loss of generality.

Suppose that $A$ can be written as $[A_1 \mid A_2]$ with $A_1 \in \mathbb{R}^{m \times k}$ and $A_2 \in \mathbb{R}^{m \times (n-k)}$ where the columns of $A_1$ are linearly independent and each column of $A_2$ is a linear combinations of the columns of $A_1$. We also write $c = \binom{c_1}{c_2}$ with $c_1 \in \mathbb{R}^k$ and $c_2 \in \mathbb{R}^{n-k}$ and consider the linear program

$$\max\{c_1^T x_1 \colon x_1 \in \mathbb{R}^k,\ A_1 x_1 \leqslant b\} \tag{3.33}$$

Since each column of $A_2$ is a linear combination of the columns of $A_1$, there exists a uniquely determined matrix $U \in \mathbb{R}^{k\times(n-k)}$ with $A_2 = A_1 \cdot U$.

**Lemma 3.5.** *The linear program* (3.33) *is feasible if and only if the linear program* (3.1) *is feasible.*

*Proof.* Suppose that $x^* = \binom{x_1^*}{x_2^*}$ is a feasible solution of (3.1), i.e., $A_1 x_1^* + A_2 x_2^* \leqslant b$. But $A_1 x_1^* + A_2 x_2^* = A_1(x_1^* + U \cdot x_2^*)$ which yields a feasible solution of (3.33). Likewise we see that any solution $x_1^*$ of $A_1 x_1 \leqslant b_1$ can be extended to a feasible solution $\binom{x_1^*}{0}$ of (3.1). □

**Lemma 3.6.** *If* (3.33) *is feasible and if* $c_2^T \neq c_1^T \cdot U$, *then* (3.1) *is unbounded.*

*Proof.* Since (3.33) is feasible, then also (3.1) is feasible. Let $x^*$ thus be a feasible solution of (3.1). Then $x^* + \mu\binom{-Uv}{v}$ is feasible for any $v \in \mathbb{R}^{n-k}$ and $\mu \in \mathbb{R}$. Let $v$ satisfy $c_2^T v \neq 0$, then $c_1^T(-U)v + c_2^T v \neq 0$ which implies that (3.1) is unbounded. □

The idea is thus to re-order the columns of $A$ such that the first $k$ columns are linearly independent and the last $n-k$ columns are linear combinations of the first $k$ columns. This also yields a matrix $U$ and we now solve the linear program (3.33) with the simplex algorithm. If it is infeasible or unbounded, then so is the linear program (3.1). Otherwise the simplex algorithm finds an optimal solution $x_1^*$ of (3.33). If $c_2^T = c_1^T \cdot U$ then this optimal solution $\binom{x_1^*}{0}$ is also an optimal solution of the linear program (3.33). This is because a feasible solution $\binom{x_1^*}{x_2^*}$ of (3.1) yields a feasible solution $x_1^* + U \cdot x_2^*$ of (3.33) with same objective value $c_1^T(x_1^* + Ux_2^*) = c_1^T x_1^* + c_2^T x_2^*$.

## Exercises

1) Show that the linear program (3.2) that is defined by a roof is always feasible.
2) Let $B$ be a roof of the linear program (3.1) and consider the system of linear equations
$$a_{i'}^T x = -1,\ a_i^T x = 0,\ i \in B \setminus \{i'\}$$
for an index $i' \in B$. Let $x^*$ be a feasible solution to the roof-linear program. Show that $x^* + \lambda \cdot v$ is also feasible for each $\lambda > 0$.
3) A polyhedron $P = \{x \in \mathbb{R}^n \colon Ax \leqslant b\}$ *contains a line*, if there exists a nonzero $v \in \mathbb{R}^n$ and an $x^* \in R^n$ such that for all $\lambda \in \mathbb{R}$, the point $x^* + \lambda \cdot v \in P$. Show that a nonempty polyhedron $P$ contains a line if and only if $A$ does not have full column-rank.
4) Prove Proposition 3.1.

5) Let $B$ be a roof with vertex $x_B^*$. Show that the set of feasible points $\{x \in \mathbb{R}^n : a_i^T x \leqslant b(i), i \in B\}$ of the roof is of the form $x_B^* + \operatorname{cone}\{r_1, \ldots, r_n\}$ for some suitable vectors $r_i \in \mathbb{R}^n$, $i = 1, \ldots, n$.
6) A cone $C \subseteq \mathbb{R}^n$ is *pointed* if it does not contain a line: There are no vectors $x \in C$, $v \in \mathbb{R}^n$ such that $x + \lambda v \in C$ for all $\lambda \in \mathbb{R}$.
   Prove the following variant of Carathéodory's theorem. Given some set $X \subseteq \mathbb{R}^n$, $|X| > n$ such that $\operatorname{cone}(X)$ is pointed. For any $x \in \operatorname{cone}(X)$, there exist at least two different subsets $X_1, X_2 \subseteq X$ with $|X_1| = |X_2| = n$ such that $x \in \operatorname{cone}(X_1) \cap \operatorname{cone}(X_2)$.
7) Consider the problem

$$
\begin{aligned}
\max \quad & z && (3.34)\\
\text{s.t.} \quad & x + 2y \leqslant -3 && (3.35)\\
& -2x - 3y \leqslant 5 && (3.36)\\
& -2x - y + 2z \leqslant -1 && (3.37)\\
& 3x + y \leqslant 2 && (3.38)\\
& x \leqslant 0 && (3.39)\\
& y \leqslant 0 && (3.40)\\
& z \leqslant 0. && (3.41)
\end{aligned}
$$

   Assume that we perform the simplex method, and at some point have the roof given by the rows (3.35), (3.40) and (3.41). Figure 7 shows the situation in the 2-dimensional subspace given by the hyperplane $z = 0$.
   Show that the simplex algorithm might not terminate, by giving a cycling sequence of roofs that might be selected by the simplex method. Explain why your sequence is valid (it is sufficient to give drawings here, you do not need to compute the roof vertices explicitly).
   *Hint: Never let* (3.41) *leave the roof. Then it is sufficient to consider the subspace as in the illustration.*

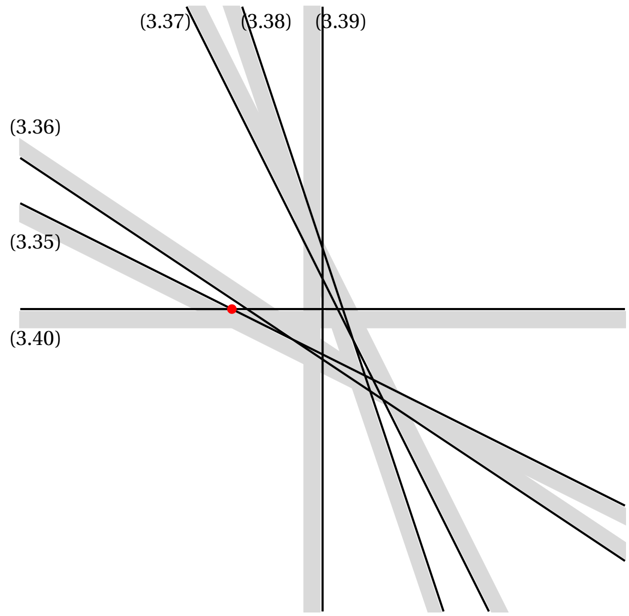


**Fig. 3.7** The halfspaces defined by system (3.34) in the subspace $\{(x, y, 0) : x, y \in \mathbb{R}\}$.

## References

1. F. Eisenbrand, A. Karrenbauer, and C. Xu. Algorithms for longer oled lifetime. In *6th International Workshop on Experimental Algorithms, (WEA 07)*, pages 338–351, 2007.
2. T. Koch. *Rapid Mathematical Programming*. PhD thesis, Technische Universität Berlin, 2004. ZIB-Report 04-58.
3. J. E. Marsden and M. J. Hoffman. *Elementary Classical Analysis*. Freeman, 2 edition, 1993.
4. J. Matouek and B. Gärtner. *Understanding and Using Linear Programming (Universitext)*. Springer-Verlag New York, Inc., Secaucus, NJ, USA, 2006.