Chapter 2
Convex sets

A polyhedron $P \subseteq \mathbb{R}^n$ is a set of the form $P = \{x \in \mathbb{R}^n \colon Ax \leqslant b\}$ for some $A \in \mathbb{R}^{m \times n}$ and some $b \in \mathbb{R}^m$. The set of feasible solutions of a linear program $\max\{c^T x \colon Ax \leqslant b\}$ is a polyhedron. Polyhedra are convex sets. Convex sets are the main objects of study of this chapter.

Linear, affine, conic and convex hulls

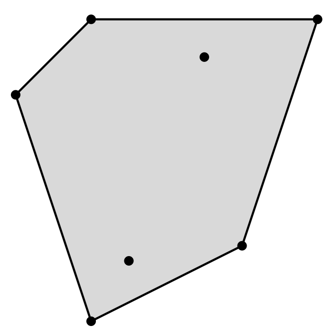

Fig. 2.1 The convex hull of 7 points in $\mathbb{R}^2$.

Let $X \subseteq \mathbb{R}^n$ be a set of n-dimensional vectors. The *linear hull, affine hull, conic hull* and *convex hull* of X are defined as follows.

$$\begin{aligned}
\text{lin.hull}(X) &= \{\lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \geqslant 0,\, x_1,\ldots,x_t \in X,\, \lambda_1,\ldots,\lambda_t \in \mathbb{R}\} && (2.1)\\
\text{affine.hull}(X) &= \{\lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \geqslant 1, && (2.2)\\
&\quad x_1,\ldots,x_t \in X,\, \sum_{i=1}^{t} \lambda_i = 1,\, \lambda_1,\ldots,\lambda_t \in \mathbb{R}\}\\
\text{cone}(X) &= \{\lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \geqslant 0, && (2.3)\\
&\quad x_1,\ldots,x_t \in X,\, \lambda_1,\ldots,\lambda_t \in \mathbb{R}_{\geqslant 0}\}\\
\text{conv}(X) &= \{\lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \geqslant 1, && (2.4)\\
&\quad x_1,\ldots,x_t \in X,\, \sum_{i=1}^{t} \lambda_i = 1,\, \lambda_1,\ldots,\lambda_t \in \mathbb{R}_{\geqslant 0}\}
\end{aligned}$$

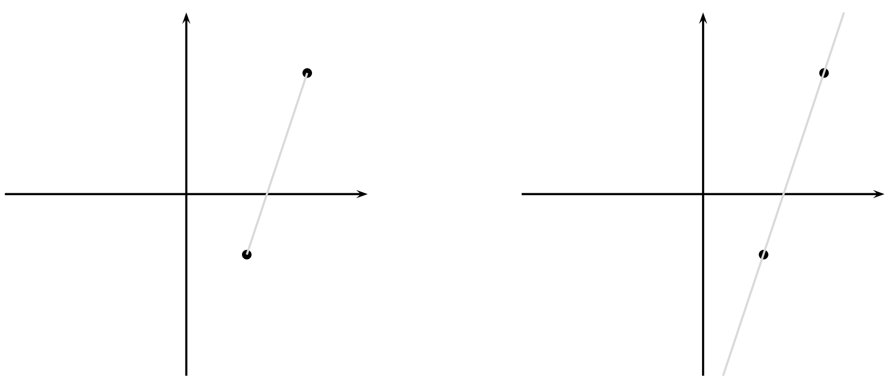

Fig. 2.2 Two points with their convex hull on the left and their affine hull on the right.

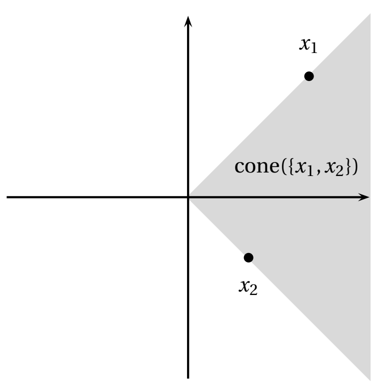

Fig. 2.3 Two points with their conic hull

Proposition 1. *Let* $X \subseteq \mathbb{R}^n$ *and* $x_0 \in X$. *One has*

$$\text{affine.hull}(X) = x_0 + \text{lin.hull}(X - x_0),$$

where for $u \in \mathbb{R}^n$ *and* $V \subseteq \mathbb{R}^n$, $u + V$ *denotes the set* $u + V = \{u + v \mid v \in V\}$.

Proof. We first show that each $x \in \text{affine.hull}(X)$ is also an element of the set $x_0 + \text{lin.hull}(X - x_0)$ and then we show that each point $x \in x_0 + \text{lin.hull}(X - x_0)$ is also an element of affine.hull(X).

Let $x \in \text{affine.hull}(X)$,i.e., there exists a natural number $t \geqslant 1$ and $\lambda_1, \dots, \lambda_t \in \mathbb{R}$, with $x = \lambda_1 x_1 + \cdots + \lambda_t x_t$ and $\sum_{i=1}^t \lambda_i = 1$. Now

$$\begin{aligned} x &= x_0 - x_0 + \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_t x_t \\ &= x_0 - \lambda_1 x_0 - \cdots - \lambda_t x_0 + \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_t x_t \\ &= x_0 + \lambda_1 (x_1 - x_0) + \cdots + \lambda_t (x_t - x_0), \end{aligned}$$

which shows that $x \in x_0 + \text{lin.hull}(X - x_0)$.

Suppose now that $x \in x_0 + \text{lin.hull}(X - x_0)$. Then there exist $\lambda_1, \dots, \lambda_t \in \mathbb{R}$ with $x = x_0 + \lambda_1 (x_1 - x_0) + \cdots + \lambda_t (x_t - x_0)$. With $\lambda_0 = 1 - \sum_{i=1}^t \lambda_i$ one has $\sum_{i=0}^t \lambda_i = 1$ and

$$\begin{aligned} x &= x_0 + \lambda_1 (x_1 - x_0) + \cdots + \lambda_t (x_t - x_0) \\ &= \lambda_0 x_0 + \cdots + \lambda_t x_t \end{aligned}$$

and thus that $x \in \text{affine.hull}(X)$. □

Definition 1. The convex hull of two distinct points $u \neq v \in \mathbb{R}^n$ is called a *line segment* and is denoted by $\overline{uv}$.

Definition 2. A set $K \subseteq \mathbb{R}^n$ is *convex* if for each $u \neq v$, the line-segment $\overline{uv}$ is contained in K, $\overline{uv} \subseteq K$.

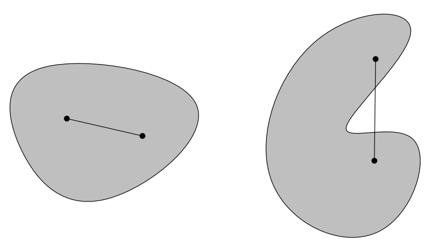

Fig. 2.4 The set on the left is convex, the set on the right is non-convex.

In other words, a set $K \subseteq \mathbb{R}^n$ is convex, if for each $u, v \in K$ and $\lambda \in [0, 1]$ the point $\lambda u + (1 - \lambda) v$ is also contained in K.

Theorem 1. *Let* $X \subseteq \mathbb{R}^n$ *be a set of points. The convex hull,* conv(X), *of* X *is convex.*

Proof. Let u and v be points in $\operatorname{conv}(X)$. This means that there exists a natural number $t \geqslant 1$, real numbers $\alpha_i, \beta_i \geqslant 0$, and points $x_i \in X$, $i = 1, \dots, t$ with $\sum_{i=1}^t \alpha_i = \sum_{i=1}^t \beta_i = 1$ with $u = \sum_{i=1}^t \alpha_i x_i$ and $v = \sum_{i=1}^t \beta_i x_i$. For $\lambda \in [0,1]$ one has $\lambda\alpha_i + (1-\lambda)\beta_i \geqslant 0$ for $i = 1, \dots, t$ and $\sum_{i=1}^t \left(\lambda\alpha_i + (1-\lambda)\beta_i\right) = 1$. This shows that

$$\lambda u + (1-\lambda) v = \sum \left(\lambda_i \alpha_i + (1-\lambda_i)\beta_i\right) x_i \in \operatorname{conv}(X),$$

and therefore that $\operatorname{conv}(X)$ is convex. □

Theorem 2. *Let $X \subseteq \mathbb{R}^n$ be a set of points. Each convex set K containing X also contains* $\operatorname{conv}(X)$.

Proof. Let K be a convex set containing X, and let $x_1, \dots, x_t \in X$ and $\lambda_i \in \mathbb{R}$ with $\lambda_i \geqslant 0$, $i = 1, \dots, t$ and $\sum_{i=1}^t \lambda_i = 1$. We need to show that $u = \sum_{i=1}^t \lambda_i x_i$ is contained in K. This is true for $t \leqslant 2$ by the definition of convex sets.

We argue by induction. Suppose that $t \geqslant 3$. If one of the λ_i is equal to 0, then one can represent u as a convex combination of $t-1$ points in X and, by induction, $u \in K$. Since $t \geqslant 3$, each $\lambda_i > 0$ and $\sum_{i=1}^t \lambda_i = 1$ one has $0 < \lambda_i < 1$ for $i = 1, \dots, t$ and thus we can write

$$u = \lambda_1 x_1 + (1-\lambda_1) \sum_{i=2}^t \frac{\lambda_i}{1-\lambda_1} x_i.$$

One has $\lambda_i/(1-\lambda_1) > 0$ and

$$\sum_{i=2}^t \frac{\lambda_i}{1-\lambda_1} = 1,$$

which means that the point $\sum_{i=2}^t \frac{\lambda_i}{1-\lambda_1} x_i$ is in K by induction. Again, by the definition of convex sets, we conclude that u lies in K. □

Theorem 2 implies that $\operatorname{conv}(X)$ is the intersection of all convex sets containing X, i.e.,

$$\operatorname{conv}(X) = \bigcap_{\substack{K \supseteq X \\ K \text{ convex}}} K.$$

Definition 3. A set $C \subseteq \mathbb{R}^n$ is a *cone*, if it is convex and for each $c \in C$ and each $\lambda \in \mathbb{R}_{\geqslant 0}$ one has $\lambda \cdot c \in C$.

Similarly to Theorem 1 and Theorem 2 one proves the following.

Theorem 3. *For any $X \subseteq \mathbb{R}^n$, the set* $\operatorname{cone}(X)$ *is a cone.*

Theorem 4. *Let $X \subseteq \mathbb{R}^n$ be a set of points. Each cone containing X also contains* $\operatorname{cone}(X)$.

These theorems imply that $\operatorname{cone}(X)$ is the intersection of all cones containing X, i.e.,

$$\operatorname{cone}(X) = \bigcap_{\substack{C \supseteq X \\ C \text{ is a cone}}} C.$$

Radon's lemma and Carathéodory's theorem

Theorem 5 (Radon's lemma). *Let $A \subseteq \mathbb{R}^n$ be a set of $n+2$ points. There exist disjoint subsets $A_1, A_2 \subseteq A$ with*

$$\operatorname{conv}(A_1) \cap \operatorname{conv}(A_2) \neq \emptyset.$$

Proof. Let $A = \{a_1, \dots, a_{n+2}\}$. We embed these points into $\mathbb{R}^{n+1}$ by appending a 1 in the $n+1$-st component, i.e., we construct

$$A' = \left\{\binom{a_1}{1}, \dots, \binom{a_{n+2}}{1}\right\} \subseteq \mathbb{R}^{n+1}.$$

The set A' consists of $n+2$ vectors in $\mathbb{R}^{n+1}$. Those vectors are linearly dependent. Let

$$0 = \sum_{i=1}^{n+2} \lambda_i \binom{a_i}{1} \tag{2.5}$$

be a nontrivial linear representation of 0, i.e., not all λ_i are 0. Furthermore, let $P = \{i \colon \lambda_i \geqslant 0,\ i = 1, \dots, n+2\}$ and $N = \{i \colon \lambda_i < 0,\ i = 1, \dots, n+2\}$. We claim that

$$\operatorname{conv}(\{a_i \colon i \in P\}) \cap \operatorname{conv}(\{a_i \colon i \in N\}) \neq \emptyset.$$

It follows from (2.5) and the fact that the $n+1$-st component of the vectors is 1 that $\sum_{i \in P} \lambda_i = -\sum_{i \in N} \lambda_i = s > 0$. It follows also from (2.5) that

$$\sum_{i \in P} \lambda_i a_i = \sum_{i \in N} -\lambda_i a_i.$$

The point $u = \sum_{i \in P} (\lambda_i / s) \cdot a_i = \sum_{i \in N} (-\lambda_i / s) a_i$ is contained in $\operatorname{conv}(\{a_i \colon i \in P\}) \cap \operatorname{conv}(\{a_i \colon i \in N\})$, implying the claim. □

Theorem 6 (Carathéodory's theorem). *Let $X \subseteq \mathbb{R}^n$, then for each $x \in \operatorname{cone}(X)$ there exists a set $\widetilde{X} \subseteq X$ of cardinality at most n such that $x \in \operatorname{cone}(\widetilde{X})$. The vectors in $\widetilde{X}$ are linearly independent.*

Proof. Let $x \in \operatorname{cone}(X)$, then there exist $t \in \mathbb{N}_+$, $x_i \in X$ and $\lambda_i \geqslant 0$, $i = 1, \dots, t$, with $x = \sum_{i=1}^t \lambda_i x_i$. Suppose that $t \in \mathbb{N}_+$ is minimal such that x can be represented as above. We claim that $t \leqslant n$. If $t \geqslant n+1$, then the x_i are linearly dependent. This means that there are $\mu_i \in \mathbb{R}$, not all equal to 0 with

$$\sum_{i=1}^{t} \mu_i x_i = 0. \tag{2.6}$$

By multiplying each μ_i in (2.6) with -1 if necessary, we can assume that at least one of the μ_i is strictly larger than 0. One has for each $\varepsilon \in \mathbb{R}$

$$x = \sum_{i=1}^{t} (\lambda_i - \varepsilon \cdot \mu_i) x_i. \tag{2.7}$$

What is the largest $\varepsilon^* > 0$ that we can pick for ε such that (2.7) is still a conic combination? We need to have

$$\lambda_i - \varepsilon \cdot \mu_i \geqslant 0, \text{ for each } i \in \{1,\dots,t\}. \tag{2.8}$$

Let J be the set of indices $J = \{j : j \in \{1,\dots,t\}, \mu_j > 0\}$. We observed that we can assume $J \neq \emptyset$. We have (2.8) as long as

$$\varepsilon \leqslant \lambda_j / \mu_j \text{ for each } j \in J. \tag{2.9}$$

This means that $\varepsilon^* = \min\{\lambda_j/\mu_j : j \in J\}$. Let $j^* \in J$ be an index where this minimum is attained. Since $\lambda_i - \varepsilon^* \cdot \mu_i \geqslant 0$ for all $i = 1,\dots,t$ and since $\lambda_{j^*} - \varepsilon^* \cdot \mu_{j^*} = 0$, we have $x \in \text{cone}(\{x_1,\dots,x_t\} \setminus \{x_{j^*}\}$, which is a contradiction to the minimality of t. □

Corollary 1 (Carathéodory's theorem for convex hulls). *Let $X \subseteq \mathbb{R}^n$, then for each $x \in \text{conv}(X)$ there exists a set $\widetilde{X} \subseteq X$ of cardinality at most $n+1$ such that $x \in \text{conv}(\widetilde{X})$.*

Separation theorem and Farkas' lemma

We recall a basic fact from analysis, see, e.g. [1, Theorem 4.4.1].

Theorem 7. *Let $X \subseteq \mathbb{R}^n$ be compact and $f : X \to \mathbb{R}$ be continuous. Then f is bounded and there exist points $x_1, x_2 \in X$ with $f(x_1) = \sup\{f(x) : x \in X\}$ and $f(x_2) = \inf\{f(x) : x \in X\}$.*

Theorem 8. *Let $K \subseteq \mathbb{R}^n$ be a closed convex set and $x^* \in \mathbb{R}^n \setminus K$, then there exists an inequality $a^T x \geqslant \beta$ such that $a^T y > \beta$ holds for all $y \in K$ and $a^T x^* < \beta$.*

Proof. Since the mapping $f(x) = \|x^* - x\|$ is continuous and since for any $k \in K$, $K \cap \{x \in K : \|x^* - x\| \leqslant \|x^* - k\|\}$ is compact, there exists a point $k^* \in K$ with minimal distance to x^*. Consider the midpoint $m = 1/2(k^* + x^*)$ on the line-segment $\overline{k^* x^*}$ and the hyperplane $a^T x = \beta$ with $\beta = a^T m$ and $a = (k^* - x^*)$. Clearly, $a^T x^* = \beta - 1/2\|k^* - x^*\|^2$ and $a^T k^* = \beta + 1/2\|k^* - x^*\|^2$. Suppose that there exists a $k' \in K$ with $a^T k' \leqslant \beta$. The points $\lambda k^* + (1-\lambda)k'$, $\lambda \in [0,1]$ are in K by the convexity of K, thus we can also assume that k' lies on the hyperplane, i.e., $a^T k' = \beta$. This means that there exists a vector x' which is orthogonal to a and $k' = m + x'$. The distance squared of a point $\lambda k^* + (1-\lambda)k'$ with $\lambda \in [0,1]$ to m is, by Pythagoras equal to

$$\lambda^2 \|\frac{1}{2} a\|^2 + (1-\lambda)^2 \|x'\|^2.$$

As a function of λ, this is increasing at at $\lambda = 1$. Thus there exists a point on the line-segment $\lambda x^* + (1-\lambda)k$ which is closer to m than k^*. This point is also closer to x^* than k^*, which is a contradiction. Therefore $a^T k > \beta$ for each $k \in K$. □

Theorem 9 (Farkas' lemma). *Let $A \in \mathbb{R}^{m\times n}$ be a matrix and $b \in \mathbb{R}^m$ be a vector. The system $Ax = b$, $x \geqslant 0$ has a solution if and only if for all $\lambda \in \mathbb{R}^m$ with $\lambda^T A \geqslant 0$ one has $\lambda^T b \geqslant 0$.*

Proof. Suppose that $x^* \in \mathbb{R}^n_{\geqslant 0}$ satisfies $Ax^* = b$ and let $\lambda \in \mathbb{R}^m_{\geqslant 0}$ with $\lambda^T A \geqslant 0$. Then $\lambda^T b = \lambda^T Ax^* \geqslant 0$, since $\lambda^T A \geqslant 0$ and $x^* \geqslant 0$.

Now suppose that $Ax = b$, $x \geqslant 0$ does not have a solution. Then, with $X \subseteq \mathbb{R}^n$ being the set of column vectors of A, b is not in $\operatorname{cone}(X)$. The set $\operatorname{cone}(X)$ is convex and closed, see exercise 5. Theorem 8 implies that there is an inequality $\lambda^T x \geqslant \beta$ such that $\lambda^T y > \beta$ for each $y \in \operatorname{cone}(X)$ and $\lambda^T b < \beta$. Since for each $a \in X$ and each $\mu \geqslant 0$ one has $\mu \cdot a \in \operatorname{cone}(X)$ and thus $\lambda^T(\mu \cdot a) > \beta$, it follows that $\lambda^T a \geqslant 0$ for each $a \in X$. Furthermore, since $0 \in \operatorname{cone}(X)$ it follows that $0 \geqslant \beta$ and thus that $\lambda^T b < 0$.

Exercises

1) Let $\{C_i\}_{i\in I}$ be a family of convex subsets of $\mathbb{R}^n$. Show that the intersection $\bigcap_{i\in I} C_i$ is convex.
2) Show that the set of feasible solutions of a linear program is convex.
3) Prove Carathéodory's Theorem for convex hulls, Corollary 1.
4) Let $A \in \mathbb{R}^{n\times n}$ be a non-singular matrix and let $a_1, \dots, a_n \in \mathbb{R}^n$ be the columns of A. Show that $\operatorname{cone}(\{a_1, \dots, a_n\})$ is the polyhedron $P = \{y \in \mathbb{R}^n \colon A^{-1} y \geqslant 0\}$. Show that $\operatorname{cone}(\{a_1, \dots, a_k\})$ for $k \leqslant n$ is the set $P_k = \{y \in \mathbb{R}^n \colon a_i^{-1} x \geqslant 0, i = 1, \dots, k,\ a_i^{-1} x = 0, i = k+1, \dots, n\}$, where a_i^{-1} denotes the i-th row of A^{-1}.
5) Prove that for a finite set $X \subseteq \mathbb{R}^n$ the conic hull $\operatorname{cone}(X)$ is closed and convex. *Hint: Use Carathéodory's theorem and exercise 4.*
6) Find a countably infinite set $X \subset \mathbb{R}^2$ such that $\operatorname{cone}(X)$ is not closed. Are there any cones that are open?
7) Prove Theorem 3.
8) Prove Theorem 4.
9) Let $f : \mathbb{R}^n \to \mathbb{R}^d$ be a linear map.
   a) Show that $f(K) = \{f(x) \colon x \in K\}$ is convex if K is convex. Is the reverse also true?
   b) For $X \subseteq \mathbb{R}^n$ arbitrary, prove that $\operatorname{conv}(f(X)) = f(\operatorname{conv}(X))$.
10) Using Theorem 9, prove the following variant of Farkas' lemma: Let $A \in \mathbb{R}^{m\times n}$ be a matrix and $b \in \mathbb{R}^m$ be a vector. The system $Ax \leqslant b$, $x \in \mathbb{R}^n$ has a solution if and only if for all $\lambda \in \mathbb{R}^m_{\geqslant 0}$ with $\lambda^T A = 0$ one has $\lambda^T b \geqslant 0$.
11) Provide an example of a convex and closed set $K \subseteq \mathbb{R}^2$ and a linear objective function $c^T x$ such that $\min\{c^T x \colon x \in K\} > -\infty$ but there does not exist an $x^* \in K$ with $c^T x^* \leqslant c^T x$ for all $x \in K$.
12) Consider the vectors

$$x_1 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, x_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, x_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, x_5 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Let $A = \{x_1, \dots, x_5\}$. Find two disjoint subsets $A_1,\ A_2 \subseteq A$ such that

$$\text{conv}(A_1) \cap \text{conv}(A_2) \neq \emptyset.$$

Hint: Recall the proof of Radon's lemma

13) Consider the vectors

$$x_1 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, x_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, x_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, x_5 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

The vector

$$v = x_1 + 3x_2 + 2x_3 + x_4 + 3x_5 = \begin{pmatrix} 15 \\ 14 \\ 25 \end{pmatrix}$$

is a conic combination of the x_i.
Write v as a conic combination using only three vectors of the x_i.
Hint: Recall the proof of Carathéodory's theorem

References

1. J. E. Marsden and M. J. Hoffman. *Elementary Classical Analysis*. Freeman, 2 edition, 1993.