Convexity

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Assignment Sheet 6

October 26, 2015

Exercise 1

Fill in the missing part of the proof of Brunn's inequality for slice volumes. That is, consider $A = C \cap \{x_1 = 0\}$, $B = C \cap \{x_1 = 1\}$ and $M = C \cap \{x_1 = \lambda\}$ for $\lambda \in (0, 1)$. Prove that

$$M \supseteq (1 - \lambda)A + \lambda B$$

Exercise 2

Let $A, B \subseteq \mathbb{R}^n$ be nonempty compact sets. For each $k \in \mathbb{N}$, consider the closed cubes of length 2^{-k} centered at the points of the scaled grid $2^{-k} \cdot \mathbb{Z}^n$. Let A_k be the union of all such cubes intersecting A, and similarly for B_k. We showed that $(A + B) \supseteq \bigcap_{k \in \mathbb{N}} (A_k + B_k)$. Show that this inclusion is in fact an equality; that is, show

$$(A + B) = \bigcap_{k \in \mathbb{N}} (A_k + B_k).$$

Exercise 3

Let $A \subseteq \mathbb{R}^n$ be a brick set containing of at least two bricks and let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$. Show that there exist $a \in \{e_1, \ldots, e_n\}$,$b \in \mathbb{R}$ and two bricks $B_1, B_2 \in A$ s.t. $a^\intercal x \leq b$ for all $x \in B_1$ and $a^\intercal x \geq b$ for all $x \in B_2$. That is, show there exists a hyperplane parallel to one of the coordinate hyperplanes that separates two bricks of A completely.

Exercise 4 [$\star$]

Let E be an equator of the unit ball B_1^n and let A_t be a belt of width $2t$ around E for $t \in (0, 1)$. Formally, $E = \{x \in S^{n-1} : a^\intercal x = 0\}$ and $A_t = \{x \in S^{n-1} : |a^\intercal x| \leq t\}$ for some $a \in \mathbb{R}^n \setminus \{0\}$.

Show that if $\Pr(A_t) = \frac{1}{2}$, then $t = O(n^{-\frac{1}{2}})$, that is, half of the measure on the sphere is concentrated in the strip of width $O(n^{-\frac{1}{2}})$ around an equator.

[*Hint:* Recall the measure concentration inequality on a sphere, that we saw on the lecture, for subset $X \in S^{n-1}$ with $\Pr(X) \geq \frac{1}{2}$. Apply this to the two halfspheres defined by the equator.]

The deadline for submitting solutions is **Monday, November 2, 2015**.