Assignment Sheet 4
October 12, 2015

Exercise 1 [⋆]
Let \( \Lambda \subset \mathbb{R}^d \) be a lattice and let \( v \in \Lambda \setminus \{0\} \) be a shortest non-zero vector of \( \Lambda \). Show there exist a basis of \( \Lambda \) of the form \((v, a_2, \ldots, a_d)\).

[Hint: Starting from a basis \((b_1, \ldots, b_d)\) of \( \Lambda \), show that there exists \( U \in \mathbb{Z}^{d \times d} \) with \( \det(U) = \pm 1 \) and \((b_1, \ldots, b_d)U = (v, a_2, \ldots, a_d)\).]

Exercise 2
Let \( \Lambda \subset \mathbb{R}^d \) be a lattice, \( v \) a shortest non-zero vector in \( \Lambda \) and \((v, a_2, \ldots, a_d)\) a basis of \( \Lambda \). Show that \( pr(\Lambda) \) is a \((d - 1)\)-dimensional lattice with basis \((pr(a_2), \ldots, pr(a_d))\), where \( pr \) is the projection map on the subspace of \( \mathbb{R}^d \) orthogonal to \( v \), i.e.

\[
pr : \mathbb{R}^d \to \mathbb{R}^d
x \mapsto \left( x - \frac{x^T v}{v^T v} v \right)
\]

Exercise 3
Let \( \Lambda \subset \mathbb{R}^2 \) be a lattice. Prove directly that \( \mu(\Lambda) \rho(\Lambda^*) \leq \sqrt{5} \).

Exercise 4
Draw ellipsoid \( E \) defined by \( A = \begin{pmatrix} 1 & 3 \\ 2 & 9 \end{pmatrix} \) and \( b = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \). What are the axes of \( E \)?

Exercise 5
Let \( K \subset \mathbb{R}^d \) be a compact convex body with a non-empty interior and suppose you are given \( E_{inr} \), the ellipsoid of largest volume contained in \( K \). Show how to compute \( u \in \mathbb{Z}^d \) s.t. \( \max_{x, y \in K} u^T(x - y) \leq d \cdot w(K) \) by one shortest lattice vector computation, where \( w(K) \) is defined to be

\[
w(K) = \min_{u \in \mathbb{Z}^d \setminus \{0\}} \max_{x, y \in K} u^T(x - y)
\]

Exercise 6
Show how to solve the following problem in \( n^{O(n)} \) calls to an oracle that computes an integer vector \( u \in \mathbb{Z}^d \setminus \{0\} \) with \( \max_{x, y \in K} u^T(x - y) \leq d \cdot w(K) \).

Given a convex body \( K \), decide whether \( K \cap \mathbb{Z}^d = \emptyset \).

The deadline for submitting solutions is **Monday, October 19, 2015**.