Assignment Sheet 1  
September 18, 2015

Exercise 1
Let $D \subseteq \mathbb{R}^d$ be a convex set and let $f : D \rightarrow \mathbb{R}$ be convex. That is, $\forall a, b \in D, \forall \lambda \in [0,1]$ 
$$f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda)f(b)$$

a) Show that the following set is convex: 
$$C = \{(x,y) : x \in D, y \geq f(x)\}$$

b) Prove or give a counterexample to the following assertion: 
If $D \subseteq \mathbb{R}^d$ is convex, $f : D \rightarrow \mathbb{R}$ and $C$ as constructed above is a convex set, then $f$ is convex.

Exercise 2
Recall the Separation theorem: 
Let $C \subseteq \mathbb{R}^d$ be closed and convex. If $x^* \notin C$, then there exists a hyperplane $a^\top x = \beta$ s.t. $a^\top x^* < \beta$ and $\forall x \in C$ it holds that $a^\top x > \beta$.

In the lecture, we proved the theorem for bounded $C$. Extend this proof for general, unbounded $C$.

Exercise 3
Give a proof of Caratheodory's theorem: 
Let $X \subseteq \mathbb{R}^d$. Then each point in conv$(X)$ is in conv$(S)$ for some $S \subseteq X, |S| \leq d + 1$.

Exercise 4 [⋆]
Let $X \subseteq \mathbb{R}^2$. For each point $x \in X$, let us denote $V(x)$ the set of all points $y \in X$ that can ”see” $x$, i.e. points s.t. the segment $xy$ is contained in $X$. More formally, for $x \in X$ let 
$$V(x) = \{y \in X : \forall \lambda \in [0,1], \lambda x + (1 - \lambda)y \in X\}$$

The kernel of $X$ is the set of all points $x \in X$ for which $V(x) = X$.

a) Prove that the kernel of any set $X \subseteq \mathbb{R}^2$ is convex.

b) Construct a nonempty set $X \subseteq \mathbb{R}^2$ such that each of its finite subsets can be seen from some point of $X$ but the kernel of $X$ is empty.

The deadline for submitting solutions is **Friday, September 25, 2015**.