Exercises marked with a ★ can be handed in for bonus points. Due date is November 19.

Exercise 1
Apply the algorithm for computing a minimum-cost \( r \)-arborescence to the digraph below. Then replace the edge of cost 1 with an edge of cost 5, and solve again.

Exercise 2
Consider the minimum disconnection problem: given a graph \( G(V,E) \) with costs \( c : E \to \mathbb{R} \), find the set \( F \subseteq E \) of minimum cost such that \( G(V,E \setminus F) \) is disconnected. Under which conditions this is equivalent to the minimum cut problem on \( G,c \)?

Exercise 3 (★)
Consider a finite set of points \( V \subseteq \mathbb{R}^2 \) not lying on the same line. Those induce a complete graph \( G \) with vertex set \( V \) and edge cost \( c(uv) = ||u - v||_2 \) for each \( u,v \in V \). The Voronoi diagram of \( V \) is the family of sets

\[
P_v = \{ x \in \mathbb{R}^2 : ||x - v||_2 = \min_{u \in V} ||x - u||_2 \}
\]

for \( v \in V \). The Delaunay triangulation of \( V \) is the graph \( G'(V,E') \) with vertex set \( V \) and edges \( uv \) for each \( u \neq v \in V \) such that \( |P_v \cap P_u| > 1 \).

(a) Show that there exists a minimum spanning tree (mst) of \( G \) whose edge set is contained in \( E' \). Is the previous true for every mst of \( G \)? Prove or give a counterexample.

(b) Suppose now to be given the adjacency lists and the incidence matrix of \( G' \), as well as the cost \( c \). How fast can a minimum spanning tree in \( G \) be computed?
(c) Now suppose that $c$ is not the euclidean distance, but any given norm. Does any of the two statement from (a) hold? Prove or give counterexamples.

**Exercise 4**
In class we saw that a submodular function is a function $f : 2^I \rightarrow \mathbb{R}$ where $I$ is a ground set, that satisfies

1. For every $S, T \subseteq I$, $f(S) + f(T) \geq f(S \cap T) + f(S \cup T)$.

(a) Show that each of the following two properties is equivalent to 1.
2. For each $S \subseteq T \subseteq I$ and $x \in I \setminus T$, we have $f(T \cup \{x\}) - f(T) \leq f(S \cup \{x\}) - f(S)$.
3. For each $S \subseteq I$ and $x \neq y \in I \setminus S$, we have $f(S \cup \{x\}) + f(S \cup \{y\}) \geq f(S \cup \{x, y\}) + f(S)$.

(b) Consider the following modification of property 2:

2'. For each $S \subseteq T \subseteq I$ and $x \in I$, we have $f(T \cup \{x\}) - f(T) \leq f(S \cup \{x\}) - f(S)$.

Show how to modify 1 to 1' as to have equivalence between 1' and 2'.

**Exercise 5**
A function is supermodular if $-f$ is submodular. For each of the following functions, detect if it is submodular, supermodular, both, or none of the two.

(a) Given a graph $G(V, E)$ with costs $c$ on the edges, let $f(S) = c(\delta(S))$ for $S \subseteq V$.
(b) Given a directed graph $D(V, A)$ with nonnegative costs $c$ on the arcs, let $f(S) = c(\delta_+(S))$ for $S \subseteq V$.
(c) Given a graph $G(V, E)$, let $f(F)$ be the number of connected components of the graph $G(V, F)$, for all $F \subseteq E$.
(d) Given a graph $G(V, E)$, let $f(S)$ be the number of connected components of the subgraph of $G$ induced by $S$ (i.e. $G'(S, E')$ with $uv \in E'$ iff $uv \in E$ and $u, v \in S$), for all $S \subseteq V$.
(e) Given a bipartite graph $G(A \cup B, E)$, let $f(X) = |N(X)|$ for all $X \subseteq A$.
(f) The rank function of a matroid.
(g) Let $X_1, \ldots, X^n$ be discrete random variables. For $S \subseteq \{1, \ldots, n\}$, let $f(S)$ be the entropy of the random variables $\{X_i : i \in S\}$.

**Exercise 6**
Deduce from the random contraction min-cut algorithm an upper bound to the number of minimum cuts in a graph. Is this bound tight?

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1. The entropy of a random variable $X$ with possible values $x_1, \ldots, x_k$ is $H(X) = -\sum_{i=1}^{k} p(x_i) \log(p(x_i))$, where $p(x_i)$ is the probability mass function of $x_i$. The entropy of a set of random variables $X_1, \ldots, X^m$, where for $i = \ldots, n$ the possible values of $X^i$ are $x^i_1, \ldots, x^i_{k_i}$, is $H(X) = -\sum_{i=1}^{n} \sum_{j=1}^{k_i} p(x^i_1, x^i_2, \ldots, x^i_{k_i}) \log(p(x^i_1, x^i_2, \ldots, x^i_{k_i}))$, where $p$ is the joint probability distribution of $X_1, \ldots, X^m$. 