Prof. F. Eisenbrand EPFL - DISOPT

Combinatorial Optimization

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Sheet 3 October 20, 2011

General remark:
In order to obtain a bonus for the final grading, you may hand in written solutions to the exercise marked with a star at the beginning of the exercise session on November 1.

Exercise 1

Let $t(x) = Ax + b$ be the mapping that transforms the 2-dimensional unit ball into the ellipsoid $E(A, b)$. Draw $E(A, b)$ for $A = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Exercise 2

(i) Show that the unit simplex $\Delta = \text{conv}\{0, e_1, \ldots, e_n\} \subset \mathbb{R}^n$, where e_j are the standard unit vectors, has volume $\frac{1}{n!}$.

(ii) Show that the volume of the simplex $\text{conv}\{v_0, v_1, \ldots, v_n\} \subset \mathbb{R}^n$ is $\frac{|\det(v_1 - v_0, \ldots, v_n - v_0)|}{n!}$.

Exercise 3 ($\star$)

The purpose of this exercise is to review the steps of the ellipsoid method in greater detail for the case where we want to use it to compute a max-weight matching of $G = (V, E)$ with positive integer weights $w \in \mathbb{Z}_+^{|E|}$.
Let μ_w be the maximum weight of a matching and let $P_G = \text{conv}\{\chi^M : M \text{ matching}\}$ be the matching polytope.
Show that

(i) $\mu_w = z^* + 1$, where z^* is the largest integer such that $P_G \cap (w^T x \geq z + \frac{1}{2})$ has positive volume.

(ii) if z is an integer such that $P_G \cap (w^T x \geq z + \frac{1}{2})$ has positive volume, then $\text{vol}\left(P_G \cap (w^T x \geq z + \frac{1}{2})\right) \geq \frac{1}{n!(2\|w\|)^n \cdot n^{n/2}}$

(iii) the ellipsoid-method needs a polynomial (in encoding length) number of iterations, to find out whether $P_G \cap (w^T x \geq z)$ has positive volume for some integer z. (Find a suitable initial ellipsoid and estimate number of iterations)

(iv) Complete the description of an algorithm using the ellipsoid method that computes μ_w (binary search) and a point in $P_G \cap (w^T x \geq \mu_w - \frac{1}{2})$. You can assume an oracle for the separation problem.

Exercise 4

Describe an algorithm that given as input

- an integral polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$,
- an objective function vector $c \in \mathbb{Z}^n$,
- the optimal objective function value $z = \max\{c^T x \mid x \in P\}$, and
- a feasible point $x^\star \in P$ with $z - \frac{1}{2} \leq c^T x^\star \leq z$

computes an inclusion-wise minimal optimal face $F = \{x \in \mathbb{R}^n \mid A'x = b'\}$ of P. The running time of your algorithm shall be bounded by a polynomial in n and m.