Exercise 1

Recall the following problems seen in class: the Steiner tree problem (st) takes as input a graph $G(V,E)$, terminals $X \subseteq V$, and costs $c : E \rightarrow \mathbb{R}_+^n$, and outputs the subgraph of $G$ of minimum cost that is connected and contains all terminals; the single source rent-or-buy problem (ssrob) takes as input a graph $G(V,E)$, terminals $X \subseteq V$, a root $r \in V$, costs $c : E \rightarrow \mathbb{R}_+^n$, and an integer $M \in \mathbb{N}$, and aims at finding, for each terminal $t \in X$, a path between $t$ and $r$, so that the total cost $f = \sum_e f_e(x_e)$ is minimized, where $x_e$ is the number of such paths containing $e$ and $f_e(x_e) = c_e \cdot \min\{x_e, M\}$.

Exercises marked with a ★ can be handed in for bonus points. Due date is November 05.

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Exercise 1

Recall the 2-approximation algorithm for solving st seen in class: Construct the closure $G'$ of $G$, i.e. the complete graph with node set $X$ and edge costs between nodes $u$ and $v$ being the shortest path from $u$ to $v$ in $G$. Take a minimum spanning tree over $G'$, and replace each edge with the corresponding shortest path, eliminating cycles if any.

1. Apply the algorithm to the graph below, where terminals are represented by squares.

![Graph](image)

2. Show that the analysis of the algorithm can be improved, as to prove that it is a $2(1 - \frac{1}{|X|})$-approximation.

3. Show that the analysis from part [2.] is tight, i.e. provide an instance where the algorithm outputs a solution whose ratio with the optimal solution is arbitrarily close to $2(1 - \frac{1}{|X|})$.

Exercise 2

In class we saw that, using the 2-approximation algorithm for st that involves the computation of a minimum spanning tree, one can obtain a 4-approximation for ssrob.
1. Apply the algorithm to the ssrob instance defined by the graph from the previous exercise, with root the leftmost node and $M = 2$ (suppose that at the first step of the algorithm, only the second leftmost node has been extracted).

2. Now suppose to be given an $\alpha$-approximate algorithm for $st$ as a black box (i.e. you do not know what the algorithm does, but only obtain its output). Could you improve the approximation algorithm for ssrob?

Exercise 3
In class we saw that the optimal solution to ssrob is always a tree. We are going to show how this can be achieved. First, along the lines of what we saw in class, prove the following:

1. If ssrob has a unique optimal solution, then it is a tree.

In order to achieve [1.] without changing the structure of the problem, prove:

2. Modify the cost of edges so that ssrob has a unique optimal solution, and it is one of the optimal solutions of the original problem. Deduce that any instance of ssrob has an optimal solution that is a tree.

More generally, we want to show that there is always an optimal solution that is a tree for the following problem, which we call generalized ssrob (gssrob): given a graph $G(V, E)$, terminals $X \subseteq V$, a root $r \in V$, concave, non-decreasing costs functions $f_e : \mathbb{R} \rightarrow \mathbb{R}_+$ for each edge $e$, connect $r$ to $X$ minimizing $f = \sum e f_e(x_e)$. This can be shown through the following steps:

3 Show that if the cost functions on the edges are also strictly concave, then all optimal solutions to $P$ are trees.

4 Show that we can modify the cost functions on the edges so that they are strictly concave and non-decreasing, and all the optimal solutions of the new problem are also optimal solutions to the original problem. Deduce that any instance of gssrob has an optimal solution that is a tree.

Exercise 4 ($\star$)
Consider the following problem: Given a set $S$ of elements, and a family $\mathcal{F} = \{F_1, \ldots, F_t\}$ of subsets of $S$ with costs $c : \mathcal{F} \rightarrow \mathbb{R}_+$, find the subfamily $\mathcal{E}$ of $\mathcal{F}$ of minimum cost such that each element from $S$ is in at least one set from $\mathcal{E}$. Consider the following algorithm:

- Set $C = \emptyset$, $\mathcal{I} = \emptyset$.

- While $C \neq S$
  
  Let $F \in \mathcal{F} \setminus \mathcal{I}$ be the set that minimizes $\frac{c(F)}{|F-C|}$, add $F$ to $\mathcal{I}$ and set $C = C \cup F$.

1. Prove that it is a $1 + 1/2 + \cdots + 1/n = \log(n) + O(1)$ approximation for the problem. (Hint: For each $e \in S$ which is covered in the output solution by a set $F_e$, set $\text{price}(e) = \frac{c(F_e)}{|F_e-C|}$. What is $\sum e \text{price}(e)$? Can you upper bound $\text{price}(e)$ in terms of the cost of the optimal solution?)

2. Provide an example showing that the approximation ratio is tight.