Prof. Friedrich Eisenbrand
Dr. Yuri Faenza

Combinatorial Optimization

Fall 2013

Assignment Sheet 3

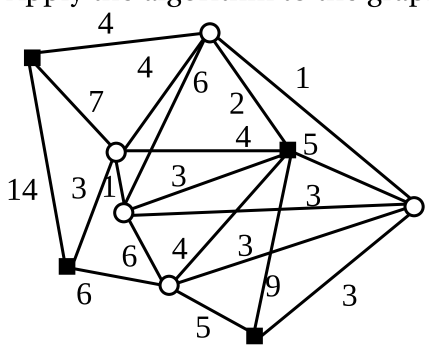
Exercises marked with a ⋆ can be handed in for bonus points. Due date is November 05.

Recall the following problems seen in class: the Steiner tree problem (st) takes as input a graph $G(V,E)$, terminals $X \subseteq V$, and costs $c : E \to \mathbb{R}^n_+$, and outputs the subgraph of G of minimum cost that is connected and contains all terminals; the single source rent-or-buy problem (ssrob) takes as input a graph $G(V,E)$, terminals $X \subseteq V$, a root $r \in V$, costs $c : E \to \mathbb{R}^n_+$, and an integer $M \in \mathbb{N}$, and aims at finding, for each terminal $t \in X$, a path between t and r, so that the total cost $f = \sum_e f_e(x_e)$ is minimized, where x_e is the number of such paths containing e and $f_e(x_e) = c_e \cdot \min\{x_e, M\}$.

Exercise 1

Recall the 2-approximation algorithm for solving st seen in class: Construct the *closure* G' of G, i.e. the complete graph with node set X and edge costs between nodes u and v being the shortest path from u to v in G. Take a minimum spanning tree over G', and replace each edge with the corresponding shortest path, eliminating cycles if any.

1. Apply the algorithm to the graph below, where terminals are represented by squares.



2. Show that the analysis of the algorithm can be improved, as to prove that it is a $2(1 - \frac{1}{|X|})$-approximation.

3. Show that the analysis from part [2.] is tight, i.e. provide an instance where the algorithm outputs a solution whose ratio with the optimal solution is arbitrarily close to $2(1 - \frac{1}{|X|})$.

Exercise 2

In class we saw that, using the 2−approximation algorithm for st that involves the computation of a minimum spanning tree, one can obtain a 4-approximation for ssrob.

1. Apply the algorithm to the ssrob instance defined by the graph from the previous exercise, with root the leftmost node and $M = 2$ (suppose that at the first step of the algorithm, only the second leftmost node has been extracted).

2. Now suppose to be given an α-approximate algorithm for st as a black box (i.e. you do not know what the algorithm does, but only obtain its output). Could you improve the approximation algorithm for ssrob?

Exercise 3
In class we saw that the optimal solution to ssrob is always a tree. We are going to show how this can be achieved. First, along the lines of what we saw in class, prove the following:

1. If ssrob has a unique optimal solution, then it is a tree.

In order to achieve [1.] without changing the structure of the problem, prove:

2. Modify the cost of edges so that ssrob has a unique optimal solution, and it is one of the optimal solutions of the original problem. Deduce that any instance of ssrob has an optimal solution that is a tree.

More generally, we want to show that there is always an optimal solution that is a tree for the following problem, which we call *generalized* ssrob (gssrob): given a graph $G(V,E)$, terminals $X \subseteq V$, a root $r \in V$, concave, non-decreasing costs functions $f_e : \mathbb{R} \to \mathbb{R}_+$ for each edge e, connect r to X minimizing $f = \sum_e f_e(x_e)$. This can be shown through the following steps:

3 Show that if the cost functions on the edges are also strictly concave, then all optimal solutions to P are trees.

4 Show that we can modify the cost functions on the edges so that they are strictly concave and non-decreasing, and all the optimal solutions of the new problem are also optimal solutions to the original problem. Deduce that any instance of gssrob has an optimal solution that is a tree.

Exercise 4 ($\star$)
Consider the following problem: Given a set S of elements, and a family $\mathscr{F} = \{F_1, \dots, F_t\}$ of subsets of S with costs $c : \mathscr{F} \to \mathbb{R}_+$, find the subfamily $\mathscr{E}$ of $\mathscr{F}$ of minimum cost such that each element from S is in at least one set from $\mathscr{E}$. Consider the following algorithm:

- Set $C = \emptyset$, $\mathscr{S} = \emptyset$.
- While $C \neq S$

  Let $F \in \mathscr{F} \setminus \mathscr{S}$ be the set that minimizes $\frac{c(F)}{|F-C|}$, add F to $\mathscr{S}$ and set $C = C \cup F$.

1. Prove that it is a $1 + 1/2 + \cdots + 1/n = \log(n) + O(1)$ approximation for the problem.
   (Hint: For each $e \in S$ which is covered in the output solution by a set F_e, set price$(e) = \frac{c(F_e)}{|F_e - C|}$. What is $\sum_e$price(e)? Can you upper bound price(e) in terms of the cost of the optimal solution?)

2. Provide an example showing that the approximation ratio is tight.