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Discussions from: October 08, 2013

Combinatorial Optimization

Fall 2013

Assignment Sheet 2

Exercises marked with a ⋆ can be handed in for bonus points. Due date is October 22.

Recall that a a pair $(S,\mathscr{I})$, with $\mathscr{I}$ a family of subsets of S, is called an *independence system* if $\emptyset \in \mathscr{I}$ and moreover $I \subseteq J \in \mathscr{I}$ implies $I \in \mathscr{I}$. In a *matroid*, we have the further condition that, if $I, J \in \mathscr{I}$ with $|J| > |I|$, then there exists $x \in J \setminus I$ such that $I \cup \{x\} \in \mathscr{I}$.

Exercise 1

All of the following are famous combinatorial optimization problems. Formulate each of them as the problem of finding a basis of minimum weight in an appropriate independence system. Investigate which of them define matroids.

1. Given a digraph $D(V,E)$ with costs $c: E \to \mathbb{R}$, and $s,t \in V$, find a shortest $s-t$ path in D with respect to c.
2. Given a connected undirected graph $G(V,E)$ weights $c: E \to \mathbb{R}_+$, and a set $T \subseteq V$ of *terminals*, find a tree $S(V',E')$ with $T \subseteq V' \subseteq V$ and $E' \subseteq E$ of minimum cost.
3. Given a complete undirected graph $G(V,E)$ and weights $c: E \to \mathbb{R}_+$, find a cycle of minimum cost that pass through all vertices of the graph.

Exercise 2

Show that $(S,\mathscr{I})$ is a matroid if and only if it is an independence system and any of the following holds.

1. if $I, J \in \mathscr{I}$ and $|J| = |I| + 1$, then $I \cup \{e\} \in \mathscr{I}$ for some $e \in J \setminus I$;
2. if $I, J \in \mathscr{I}$ and $|I \setminus J| = 1$, $|J \setminus I| = 2$, then $I \cup \{e\} \in \mathscr{I}$ for some $e \in J \setminus I$.
3. for all $A \subseteq S$, every maximal subset $I \subseteq A$ with $I \in \mathscr{I}$ has the same cardinality.

Exercise 3

Let $G = (V,E)$ be a graph. Let $\mathscr{I} \subseteq 2^V$ be defined as follows:
For $U \subseteq V$, we have $U \in \mathscr{I}$ if and only if there exists a matching in G that covers U (and possibly other vertices).
Show that $M = (V,\mathscr{I})$ is a matroid.

Exercise 4
Given matroids $M_1 = (S_1, \mathscr{I}_1)$, and $M_2 = (S_2, \mathscr{I}_2)$ with $S_1 \cap S_2 = \emptyset$, their *disjoint union* is given by $M = (S, \mathscr{I})$ with $S = S_1 \cup S_2$ and $\mathscr{I} = \{J_1 \cup J_2 : J_1 \in \mathscr{I}_1, J_2 \in \mathscr{I}_2\}$. Prove that M is a matroid, and describe its rank function.

Exercise 5
Let E be a finite set that is partitioned into sets $E = E_1 \cup \ldots \cup E_r$ and define

$$\mathscr{I} := \{S \subset E \mid |S \cap E_j| \leq 1 \text{ for all } j = 1 \ldots r\}.$$

Show that $(E, \mathscr{I})$ is a matroid. What is the rank of this matroid? Give a simple description of the bases of the matroid.
Remark: This type of matroid is called a *partition matroid*.

Exercise 6 (⋆)
In class we saw that the greedy algorithm always outputs a maximum-weight independent set of a matroid wrt any cost function c. Show that, if $(S, \mathscr{I})$ is an independence system that is *not* a matroid, then there exists a cost function $c : S \to \mathbb{R}_+$ such that the greedy algorithm does not find a maximum-weight independent set of $(S, \mathscr{I})$ wrt c.

Exercise 7
Recall that a *circuit* of a matroid is a minimal dependent set. Let $(S, \mathscr{I})$ be a matroid, let $J \in \mathscr{I}$, and $x \in S$. Then $J \cup \{x\}$ contains at most a circuit.

Exercise 8 (⋆)
Let $M = (S, \mathscr{I})$ be a matroid. Prove that $M^* = (S, \mathscr{I}^*)$ is also a matroid, where $\mathscr{I}^* = \{J \subseteq S : r(S \setminus J) = r(S)\}$. Which is its rank function?