École Polytechnique Fédérale de Lausanne
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Discrete Optimization (Spring 2018)

Assignment 13

Problem 1

Let $G = (V, E)$ be a bipartite graph and consider the perfect matching polytope of G, defined as: $Q(G) = \{x \in \mathbb{R}^E : \sum_{e:\in\delta(v)} x_e = 1 \quad \forall v \in V,\ x_e \geq 0 \quad \forall e \in E\}$. Prove that $Q(G)$ is integral, i.e. that each vertex of $Q(G)$ has integer coordinates.

Solution:
We already proved in class that the matching polytope of G, i.e. $P(G) = \{x \in \mathbb{R}^E : \sum_{e:\in\delta(v)} x_e \leq 1 \quad \forall v \in V,\ x_e \geq 0 \quad \forall e \in E\}$, is integer. But $Q(G)$ is a face of $P(G)$ (it is obtained from $P(G)$ by intersecting it with hyperplanes corresponding to valid inequalities) hence its vertices are also vertices of $P(G)$, hence they all have integer coordinates.

Problem 2

Let $G = (V, E)$ be a 4-regular bipartite graph and $w : E \to \mathbb{R}$. Prove that there exists a perfect matching in G of the weight at most $\frac{1}{4}\sum_{e\in E} w_e$.
Hint: Use Problem 1.

Solution: Let $x \in \mathbb{R}^E$ with $x_e = 1/4$ for all $e \in E$. It is a feasible solution to $Q(G)$ as defined in Problem 1 and it has the weight $\sum_{e\in E} \frac{1}{4} w_e$. Since $Q(G)$ is a polytope, there exists a vertex which attains its minimum with respect to the weights w, denote it with q. We get that

$$\sum_{e\in E} q_e w_e \leq \sum_{e\in E} \frac{1}{4} w_e = \frac{1}{4}\sum_{e\in E} w_e.$$

Furthermore, by Problem 1, q is integer, hence it is the characteristic vector of a perfect matching in G. We can conclude that there is a perfect matching in G of size at most $\frac{1}{4}\sum_{e\in E} w_e$.

Problem 3

Recall that, given a graph $G(V, E)$, a vertex cover of G is a subset $C \subset V$ such that for every edge $e \in E$, e has at least one endpoint in C. Consider the following algorithm, called Greedy, for finding a (not necessarily minimum) vertex cover in a graph G.

Greedy (V, E):

$C = \emptyset$
while $E \neq \emptyset$ **do:**
    Select any $e = \{u, v\} \in E$
    $C := C \cup \{u, v\}$
    $E := E \setminus (\delta(u) \cup \delta(v))$ % Remove all edges incident to u or v
return C

(a) Show that Greedy is correct (i.e. that it outputs a vertex cover of G). What is its asymptotic running time in terms of $|V|, |E|$? Justify your answer. (You can assume that selecting an edge and removing an edge takes constant time).

(b) Let C^* be a vertex cover of G of minimum cardinality, and let C be the vertex cover output by the Greedy algorithm. Show that $|C| \leq 2|C^*|$.

Solution:

(a) At each iteration, we add two vertices to C and remove from E all the edges that have one or both of them as endpoints (these edges are "covered" by C). Since the algorithm only stops when there is no edge left in E, C is a vertex cover for G. To compute the running time of Greedy, one could just multiply the number of iterations ($O(|E|)$) by the number of operations per iteration ($O(|E|)$). However, we notice that each edge is selected or removed only once during the algorithm, and similarly each vertex is added to C at most once, hence the total running time of Greedy is $O(|E|)$.

(b) Let $e_1, \ldots, e_k$ be the edges selected during the execution of Greedy. We have $|C| = 2k$, and $e_1, \ldots, e_k$ form a matching. Hence, since any vertex cover must cover $e_1, \ldots, e_k$ in particular, we have $|C^*| \geq k$ (we need a different vertex for each edge in the matching), which implies that $2k = |C| \leq 2|C^*|$.

Problem 4

Prove that the rank of the Tutte matrix of G is twice the size of a maximum matching in G (the "rank" here refers to largest r such that there is a $r \times r$ submatrix whose determinant is not the zero polynomial).

Hint: Let A be an $n \times n$ skew symmetric matrix (i.e. $A^\top = -A$) of rank r. For any two sets $S, T \subseteq [n]$ we denote by A_{ST} the submatrix of A indexed by rows S and columns T. For any two sets S, T of size r show that $\det(A_{ST})\det(A_{TS}) = \det(A_{TT})\det(A_{SS})$.

Solution:

We first prove the statement given in the hint. We will assume that $S \neq T$, since the statement is trivial otherwise.

- Suppose that A_{TT} has full rank r. Then, $A_{(T\cup S)T}$ has column rank r. Now, suppose that there was a column in $A_{(T\cup S)S}$, which would be inde- pendent from the columns of $A_{(T\cup S)T}$. Then, we could fi

  nd a submatrix of size $(r+1)\times(r+1)$ with $r+1$ independent columns, which contradicts the fact that A has rank r. Thus, there must exist a matrix P, such that $A_{(T\cup S)S} = A_{(T\cup S)T}P$. Then, we have $A_{TS} = A_{TT}P$ and $A_{SS} = A_{ST}P$. By skew symmetry, we also have $A_{ST} = -A_{TS}$. Combining these we get

  $$\det(A_{ST})\cdot\det(A_{TS}) = \det(A_{TS})\cdot\det(A_{TT}\cdot P) = \det(-P\cdot A_{TS})\cdot\det(A_{TT}) = \det(A_{SS})\cdot\det(A_{TT}).$$

- Now suppose that A_{TT} has rank less than r. If A_{ST} has rank less than r we are done (both sides in the statement are 0). So suppose A_{ST} has rank r. By skew symmetry, A_{TS} also has rank r. Again, we know that we can write $A_{TT} = A_{TS} \cdot P$, where P is singular. Using the same argument as previously, we can show that we must have $A_{ST} = A_{SS} \cdot P$, or A would have rank greater than r. But then,

  $$\det(A_{ST}) = \det(A_{SS} \cdot P) = \det(A_{SS})\det(P) = 0,$$

  since P is singular. This contradicts our assumption that A_{ST} had full rank.

Suppose the maximum matching is of size m, and let T be the set of matched vertices ($|T| = 2m$). We easily see that A_{TT} is simply the Tutte matrix of the graph G restricted to the vertices T. But this graph contains a perfect matching so we know, by Tutte's theorem, that $\det(A_{TT}) = 0$. Furthermore, A_{TT} is a submatrix of A of rank $2m$. Thus, we must have $r \geq 2m$. Now, suppose the rank of A is r. Thus, there must be a sub matrix of A with non zero determinant. Let this matrix be A_{TS} for some sets T and S of size r. From the result we proved previously, we get

$$\det(A_{ST}) \det(A_{TS}) = \det(A_{TT}) \det(A_{SS}) \neq 0;$$

where we use the fact that A_{ST} has the same rank as A_{TS} by skew- symmetry. Now, A_{TT} is the Tutte matrix of the graph restricted to T, which then contains a perfect matching of size $r/2$. Thus, the maximal matching in the graph satis
es $m = r/2$.

Alternative solution:
Let r be the rank of A_G, the Tutte matrix of G, and m the size of a maximum matching of G. We first show that $r \geq 2m$.
Consider a matching M with m edges, we can rename the vertices such that vertices in M get labels $1, 2, \ldots, 2m$. Consider the corresponding Tutte matrix of G, the principal submatrix made of the first $2m$ rows and columns corresponds to M and so its determinant is non-zero: this implies that the Tutte matrix has rank at least $2m$.
Suppose now $r > 2m$, i.e., there is a submatrix A_{ST}, where $S, T \subset [n]$ have size r, whose determinant is non-zero. We show that this implies that A_{SS} has non-zero determinant, therefore that S is a matching, a contradiction since S has more than $2m$ vertices. We proceed in a similar way as in the proof of Tutte's result seen in the lecture. Of course if $S = T$ we are done, so we assume $S \neq T$.The determinant A_{ST} can be seen as a sum of products of the elements of A_{ST}, each product indexed by a bijection from S to T (which correspond to permutations when $S = T$). As seen in the lecture, the only non-zero products occur when each element is non-zero, i.e., each term corresponds to an edge of G. Each product, as a bijection, is a collection of disjoint directed cycles and paths (while a permutation is made of cycles only). Any vertex in a cycle or inside a path (so any vertex of total degree 2) is in $S \cap T$, each starting point of a path is in S, each endpoint is in T. Exactly as in the proof given in the lecture, we prove that among these bijections there is one where each cycle is even, so then we have our matching in S taking alternating edges in the cycles and in each path (starting from the first edge). We pair all the bijections with odd cycles in a way that they cancel out, then we have that, since the sum over all products is non-zero, there must be a non-zero product with even cycles only. For any bijection σ, define a friend $f(\sigma)$ by reverting the order of the odd cycle that contains the vertex with high index (as in the Tutte's proof). This partitions the bijections with odd cycles in pairs and trivially the products given by σ and $f(\sigma)$ cancel out (they have opposite sign and same elements), so the whole sum of bijections with odd cycles is zero and the proof is complete.