Problem 1
Let $G = (V, E)$ be a bipartite graph and consider the perfect matching polytope of $G$, defined as:

$$Q(G) = \{ x \in \mathbb{R}^E : \sum_{e \in \delta(v)} x_e = 1 \quad \forall v \in V, \ x_e \geq 0 \quad \forall e \in E \}.$$  
Prove that $Q(G)$ is integral, i.e. that each vertex of $Q(G)$ has integer coordinates.

Problem 2
Let $G = (V, E)$ be a 4-regular bipartite graph (i.e. $|\delta(v)| = 4$ for each $v \in V$) and $w : E \to \mathbb{R}$. Prove that there exists a perfect matching in $G$ of the weight at most $\frac{1}{4} \sum_{e \in E} w_e$.

*Hint: Use Problem 1.*

Problem 3
Recall that, given a graph $G(V, E)$, a vertex cover of $G$ is a subset $C \subset V$ such that for every edge $e \in E$, $e$ has at least one endpoint in $C$. Consider the following algorithm, called Greedy, for finding a (not necessarily minimum) vertex cover in a graph $G$.

Greedy $(V, E)$:

- $C = \emptyset$
- while $E \neq \emptyset$ do:
  - Select any $e = \{u, v\} \in E$
  - $C := C \cup \{u, v\}$
  - $E := E \setminus (\delta(u) \cup \delta(v))$ % Remove all edges incident to $u$ or $v$
- return $C$

(a) Show that Greedy is correct (i.e. that it outputs a vertex cover of $G$). What is its asymptotic running time in terms of $|V|, |E|$? (You can assume that selecting an edge and removing an edge takes constant time).

(b) Let $C^*$ be a vertex cover of $G$ of minimum cardinality, and let $C$ be the vertex cover output by the Greedy algorithm. Show that $|C| \leq 2|C^*|$. 

Problem 4
Prove that the rank of the Tutte matrix of $G$ is twice the size of a maximum matching in $G$ (the “rank” here refers to largest $r$ such that there is a $r \times r$ submatrix whose determinant is not the zero polynomial).

*Hint: Let $A$ be an $n \times n$ skew symmetric matrix (i.e. $A^T = -A$) of rank $r$. For any two sets $S, T \subseteq [n]$ we denote by $A_{ST}$ the submatrix of $A$ indexed by rows $S$ and columns $T$. For any two sets $S, T$ of size $r$ show that $\det(A_{ST}) \det(A_{TS}) = \det(A_{TT}) \det(A_{SS})$. 
