École Polytechnique Fédérale de Lausanne
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Discrete Optimization (Spring 2018)

Assignment 13

Problem 1

Let $G = (V, E)$ be a bipartite graph and consider the perfect matching polytope of G, defined as: $Q(G) = \{x \in \mathbb{R}^E : \sum_{e:\in\delta(v)} x_e = 1 \quad \forall v \in V,\ x_e \geq 0 \quad \forall e \in E\}$. Prove that $Q(G)$ is integral, i.e. that each vertex of $Q(G)$ has integer coordinates.

Problem 2

Let $G = (V, E)$ be a 4-regular bipartite graph (i.e. $|\delta(v)| = 4$ for each $v \in V$) and $w : E \to \mathbb{R}$. Prove that there exists a perfect matching in G of the weight at most $\frac{1}{4}\sum_{e\in E} w_e$.
Hint: Use Problem 1.

Problem 3

Recall that, given a graph $G(V, E)$, a vertex cover of G is a subset $C \subset V$ such that for every edge $e \in E$, e has at least one endpoint in C. Consider the following algorithm, called Greedy, for finding a (not necessarily minimum) vertex cover in a graph G.

Greedy (V, E):

$C = \emptyset$
while $E \neq \emptyset$ **do:**
  Select any $e = \{u, v\} \in E$
  $C := C \cup \{u, v\}$
  $E := E \setminus (\delta(u) \cup \delta(v))$ % Remove all edges incident to u or v
return C

(a) Show that Greedy is correct (i.e. that it outputs a vertex cover of G). What is its asymptotic running time in terms of $|V|, |E|$? (You can assume that selecting an edge and removing an edge takes constant time).

(b) Let C^* be a vertex cover of G of minimum cardinality, and let C be the vertex cover output by the Greedy algorithm. Show that $|C| \leq 2|C^*|$.

Problem 4

Prove that the rank of the Tutte matrix of G is twice the size of a maximum matching in G (the "rank" here refers to largest r such that there is a $r \times r$ submatrix whose determinant is not the zero polynomial).
Hint: Let A be an $n \times n$ skew symmetric matrix (i.e. $A^T = -A$) of rank r. For any two sets $S, T \subseteq [n]$ we denote by A_{ST} the submatrix of A indexed by rows S and columns T. For any two sets S, T of size r show that $\det(A_{ST})\det(A_{TS}) = \det(A_{TT})\det(A_{SS})$.