École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand

May 24, 2018

Discrete Optimization (Spring 2018)

Assignment 12

Problem 4 can be **submitted** until June 1, 12:00 noon, into the box in front of MA C1 563.
You are allowed to submit your solutions in groups of at most three students.

Problem 1

Given a graph $G = (V, E)$, a subset $S \subseteq V$ is called *stable* (or *independent*) if $|e \cap S| \leq 1$ for each $e \in E$. Assuming that G is bipartite, show that one can find a stable set of maximum cardinality in polynomial time.

Hint: Consider the family of stable sets in a bipartite graph $G = (V, E)$, and let P be the convex hull of the corresponding indicator vectors. Describe P as $\{x \in \mathbb{R}^{|V|} : \ Ax \leq \mathbf{1},\ x \geq 0\}$, where $A \in \mathbb{Z}^{m \times |V|}$ is totally unimodular and m is polynomial in $|V|$.

Solution:
An LP relaxation to the above problem is given as:

$$\max \quad \sum_{v \in V} x_v \tag{1}$$

$$\text{s.t.} \quad x_u + x_v \leq 1 \qquad \forall\, (u, v) \in E \tag{2}$$

$$x_v \geq 0 \qquad \forall v \in V. \tag{3}$$

The matrix A, corresponding to the above LP, is the transpose of the node-edge incidence matrix of a bipartite graph, and thus it is TU. Here, we are using the fact that A is TU if and only if A^T is TU. $A \in \mathbb{Z}^{m \times n}$ being TU and $b = \mathbf{1} \in \mathbb{Z}^m$ implies that the corresponding polytope P is integral. Thus, each vertex $\bar{x} \in \mathbb{R}^{|V|}$ of P is integral, and by (2) and (3), one has $\bar{x} \in \{0, 1\}^{|V|}$. Each such vector can be interpreted as a characteristic (i.e., indicator) vector of a set $S \subseteq V$. Since $\bar{x}$ satisfies (2), the corresponding S is a stable set. Vice versa, on can verify that the characteristic vector of each stable set belongs to P.

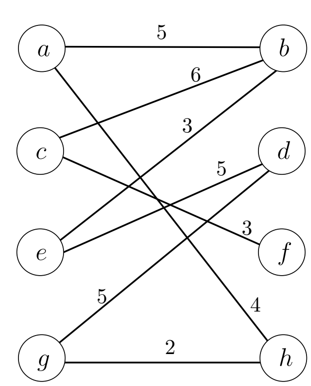
Observe that $m = |E| = O(|V|^2)$. We can solve the above LP and find an optimal vertex in time polynomial in $|V|$, by using, e.g., the ellipsoid method. Finally, the optimal vertex gives a stable set of maximal cardinality in G.

Problem 2

Given the weighted graph on the right, find the following:

a) A matching that is not perfect and has weight 15.

b) A w-vertex cover of weight 16 where at least 7 vertices have non-zero weights.



Solution:

a) $M = \{ah, cb, ed\}$.

b) One possibility is to start with the w-vertex cover (2 3 3 5 0 0 0 2) for the vertices $a - h$ respectively and transform it into (3 2 4 4 1 0 1 1).

Problem 3

Consider a graph $G = (V, E)$. A matching $M \subseteq E$ is said to be *maximal* if there is no edge $e \in E \backslash M$ such that $M \cup e$ is a matching. Denote with M^* a maximum cardinality matching in G.

a) Show that $|M| \geq \frac{|M^*|}{2}$ for any maximal matching M in G.

b) Provide a graph containing a maximal matching M with $|M| = \frac{|M^*|}{2}$.

Solution:

a) Consider an edge $\{u, v\} \in M^*$. Either $\{u, v\} \in M$ or at least one of u, v is contained in an edge of M. Hence, M covers at least $|M^*|$ vertices, i.e., it has at least $\frac{|M^*|}{2}$ edges.

b) As an example one can take a path of length 3, with the edge in the middle as a maximal matching.

Problem 4 ($\star$)

A 2-matching in a graph is a collection of disjoint cycles that covers all the vertices. Show that a 2-matching can be computed in polynomial time, if such one exists. Note that it is allowed to pick an edge twice in a 2-matching, i.e., one can have a 2-cycle.

Hint: One may reduce the problem to finding a perfect matching in a bipartite graph.

Solution:

Given the initial graph $G(V, E)$, construct a bipartite graph $G' = (V \cup V', E')$, where V' is a copy of V and E' has the edges $\{u, v'\}$ and $\{u', v\}$ for each edge $\{u, v\} \in E$.

Now, every 2-matching $M_{(2)}$ in G corresponds to a perfect matching M' in G'. For every cycle $v_1, \ldots, v_k$ in $M_{(2)}$ put the edges $(v_1, v_2'), \ldots, (v_{k-1}, v_k'), (v_k, v_1')$ in M'. Clearly, M' matches all the vertices $v_1, \ldots, v_k, v_1', \ldots, v_k'$. Doing this for all cycles in matching $M_{(2)}$ of G yields the corresponding perfect matching M' of G'.

Conversely, if there is a perfect matching M' in G', we can construct a 2-matching $M_{(2)}$ in G. For each of edges $\{u, v'\}$ and $\{u', v\}$ in M', we add the edge $\{u, v\}$ to $M_{(2)}$. A perfect matching in G' can be computed, or it can be detected that there is no such matching, in polynomial time.

Problem 5

Prove Hall's theorem: Let $G = (A \cup B, E)$ be a bipartite graph, and for each $S \subseteq A$, let

$$N(S) = \{v \in B : \exists u \in S \text{ such that } \{u, v\} \in E\}.$$

Then, G has a matching of size $|A|$ if and only if $|N(S)| \geq |S|$ for all $S \subseteq A$.

Solution:

($\Rightarrow$) If there is a matching M of size $|A|$, then for each $S \subseteq A$ there is a set $T \subseteq B$ corresponding to the neighbors of S in M. Thus, $|N(S)| \geq |T| = |S|$.

($\Leftarrow$) If there is no matching of size $|A|$, then by Köning's theorem there is a vertex cover $U \subseteq A \cup B$ such that $|U| < |A|$. Since U is a cover, we have that $N(A \setminus U) \subseteq B \cap U$, and therefore:

$$|N(A \setminus U)| \leq |B \cap U| = |U| - |A \cap U| < |A| - |A \cap U| = |A \setminus U|.$$