École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand December 23, 2016
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Combinatorial Optimization (Fall 2016)

Assignment 12

Problem 1

Let C be the square with corners at $(1,1),(1,-1),(-1,1),(-1,-1)$. Draw the polar of C. Now consider the square C' with corners at $(1,1),(1,0),(0,1),(0,0)$ and draw its polar. Why is it different from the previous one?

Solution:
The polar of C is a diamond with corners at $(1,0),(0,1),(-1,0),(0,-1)$. The polar of C' is the polyhedron described by $\{x \leq 1, y \leq 1, x+y \leq 1\}$, which is unbounded. The two polars are "different" because C' does not contain the origin in its interior, in particular the vertex $(0,0)$ does not give any constraint for the polar of C'.

Problem 2

Let $P \subseteq \mathbb{R}^n$ be a full dimensional polytope that contains the origin $\mathbf{0}$ in its interior. Let $x \in \mathbb{R}^n$. Prove that x is a vertex of P if and only if $\{y \in \mathbb{R}^n | x^\top y \leq 1\}$ defines a facet of P^0. (Hint: use the fact that any point in P can be expressed as a convex combination of the vertices in P).

Solution:
Let $Q = \{y \in \mathbb{R}^n : x^\top y \leq 1 \, \forall \, x \in V\}$ were V is the set of vertices of P.

($\Leftarrow$) This is equivalent to showing that $P^0 = Q$. Clearly, $P^0 \subseteq Q$ hence it remains to show that $Q \subseteq P^0$. Let $y \in Q$ and $x \in P$. Then x can be written as the convex combination of the vertices of P and we obtain that $x^\top y \leq 1$ as required.

($\Rightarrow$) Let x be a vertex of P. Since we showed that $P^0 = Q$ it suffices to show that $\{y \in \mathbb{R}^n : x^\top y \leq 1\}$ is facet defining for Q. Assume it is not. Then the constraints $x^\top y \leq 1$ are implied by the constraints $z^\top y \leq 1$ for vertices $z \in V \setminus \{x\}$. However, this implies that x is a convex combination of the other vertices of P, which is a contradiction.

Problem 3

Let $M(E, \mathcal{I})$ be a matroid with rank function r, $x^* \in R^{|E|}$, $f : 2^E \to \mathbb{R}$ defined as $f(X) = r(X) - x^*(X)$. Prove that f is submodular. Deduce that one can solve the separation problem for the matroid polytope P_M by solving the problem of minimizing a submodular function.

Solution:
We saw that the rank of a matroid is a submodular function, and $x^*(A)+x^*(B) = x^*(A \cup B)+x^*(A \cap B)$ for any $A, B \subset E$ (hence both $x^*(X)$ and $-x^*(X)$ are submodular). Now one can just verify the definition of submodularity, or prove more generally that any non-negative linear combinations of submodular functions is submodular. Now, given $x^* \in R^{|E|}$, we can first check in linear time whether any component x_i^* is negative, in which case clearly $x^* \notin P_M$ and $x_i = 0$ is a separation hyperplane. Otherwise we find $A^* \subset E$ such that $f(A^*)$ is minimum. If $f(A^*) \geq 0$, then clearly x^* satisfies all the inequalities $x^*(A) \leq r(A)$ for any $A \subset E$ hence $x^* \in P_M$. Otherwise, $x(A^*) = r(A^*)$ is a separation hyperplane.

Problem 4

We are given a graph $G(V, E)$ and a coloring of its edges. We want to find whether there exists a spanning tree T of G such that no two edges of T have the same color. Show how to solve this problem in polynomial time in $|V|$.

Solution:

We solve the problem as a matroid intersection problem, which is polynomial time solvable when the number of matroids considered is 2. Assume that G is colored with k colors and let $C_1, \ldots, C_k$ be the color classes of G, i.e. the set of edges with a given color. Let $M_1(E, \mathcal{I}_1)$ the forest matroid on G, and $M_2(E, \mathcal{I}_2)$ with $\mathcal{I}_2 = \{A \subset E : |A \cap C_i| \leq 1 \; \forall \, i = 1, \ldots, k\}$, which is a partition matroid. Clearly the desired spanning tree exists if and only if the largest set in $\mathcal{I}_1 \cap \mathcal{I}_2$ has size $|V| - 1$.