Problem 1
Let $C$ be the square with corners at $(1,1), (1,-1), (-1,1), (-1,-1)$. Draw the polar of $C$. Now consider the square $C'$ with corners at $(1,1), (1,0), (0,1), (0,0)$ and draw its polar. Why is it different from the previous one?

Solution:
The polar of $C$ is a diamond with corners at $(1,0), (0,1), (-1,0), (0,-1)$. The polar of $C'$ is the polyhedron described by $\{x \leq 1, y \leq 1, x + y \leq 1\}$, which is unbounded. The two polars are “different” because $C'$ does not contain the origin in its interior, in particular the vertex $(0,0)$ does not give any constraint for the polar of $C'$.

Problem 2
Let $P \subseteq \mathbb{R}^n$ be a full dimensional polytope that contains the origin $0$ in its interior. Let $x \in \mathbb{R}^n$. Prove that $x$ is a vertex of $P$ if and only if $\{y \in \mathbb{R}^n \mid x^\top y \leq 1\}$ defines a facet of $P^\circ$. (Hint: use the fact that any point in $P$ can be expressed as a convex combination of the vertices in $P$).

Solution:
Let $Q = \{y \in \mathbb{R}^n \mid x^\top y \leq 1 \forall x \in V\}$ were $V$ is the set of vertices of $P$.

$(\Rightarrow)$ This is equivalent to showing that $P^0 = Q$. Clearly, $P^0 \subseteq Q$ hence it remains to show that $Q \subseteq P^0$. Let $y \in Q$ and $x \in P$. Then $x$ can be written as the convex combination of the vertices of $P$ and we obtain that $x^\top y \leq 1$ as required.

$(\Leftarrow)$ Let $x$ be a vertex of $P$. Since we showed that $P^0 = Q$ it suffices to show that $\{y \in \mathbb{R}^n \mid x^\top y \leq 1\}$ is facet defining for $Q$. Assume it is not. Then the constraints $x^\top y \leq 1$ are implied by the constraints $z^\top y \leq 1$ for vertices $z \in V \setminus \{x\}$. However, this implies that $x$ is a convex combination of the other vertices of $P$, which is a contradiction.

Problem 3
Let $M(E, \mathcal{I})$ be a matroid with rank function $r$, $x^* \in \mathbb{R}^{|E|}$, $f : 2^E \rightarrow \mathbb{R}$ defined as $f(X) = r(X) - x^*(X)$. Prove that $f$ is submodular. Deduce that one can solve the separation problem for the matroid polytope $P_M$ by solving the problem of minimizing a submodular function.

Solution:
We saw that the rank of a matroid is a submodular function, and $x^*(A) + x^*(B) = x^*(A \cup B) + x^*(A \cap B)$ for any $A, B \subseteq E$ (hence both $x^*(X)$ and $-x^*(X)$ are submodular). Now one can just verify the definition of submodularity, or prove more generally that any non-negative linear combinations of submodular functions is submodular. Now, given $x^* \in \mathbb{R}^{|E|}$, we can first check in linear time whether any component $x_i^*$ is negative, in which case clearly $x^* \notin P_M$ and $x_i = 0$ is a separation hyperplane. Otherwise we find $A^* \subseteq E$ such that $f(A^*)$ is minimum. If $f(A^*) \geq 0$, then clearly $x^*$ satisfies all the inequalities $x^*(A) \leq r(A)$ for any $A \subseteq E$ hence $x^* \in P_M$. Otherwise, $x(A^*) = r(A^*)$ is a separation hyperplane.
Problem 4
We are given a graph $G(V,E)$ and a coloring of its edges. We want to find whether there exists a spanning tree $T$ of $G$ such that no two edges of $T$ have the same color. Show how to solve this problem in polynomial time in $|V|$.

Solution:
We solve the problem as a matroid intersection problem, which is polynomial time solvable when the number of matroids considered is 2. Assume that $G$ is colored with $k$ colors and let $C_1, \ldots, C_k$ be the color classes of $G$, i.e. the set of edges with a given color. Let $M_1(E, I_1)$ the forest matroid on $G$, and $M_2(E, I_2)$ with $I_2 = \{ A \subseteq E : |A \cap C_i| \leq 1 \ \forall \ i = 1, \ldots, k \}$, which is a partition matroid. Clearly the desired spanning tree exists if and only if the largest set in $I_1 \cap I_2$ has size $|V| - 1$. 