École Polytechnique Fédérale de Lausanne
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May 26, 2017

Discrete Optimization (Spring 2017)

Assignment 12

Problem 1

Consider the following algorithm, which takes as input a list of numbers $a_1, \dots, a_n$ and sorts them in increasing order:

Sort($a_1, \dots, a_n$):

if $n > 1$:
  return Merge(Sort($a_1, \dots, a_{\lfloor \frac{n}{2} \rfloor}$), Sort($a_{\lceil \frac{n}{2} \rceil}, \dots, a_n$))
else:
  return $a_1, \dots, a_n$

Where Merge is the following algorithm:

Merge($x_1, \dots, x_k; y_1, \dots, y_l$):

if $k = 0$: **return** $y_1, \dots, y_l$
if $l = 0$: **return** $x_1, \dots, x_k$
if $x_1 \leq y_1$:
  return x_1, Merge($x_2, \dots, x_k; y_1, \dots, y_l$)
else:
  return y_1, Merge($x_1, \dots, x_k; y_2, \dots, y_l$)

1. Modify the algorithm so that it sorts the number in decreasing order instead.
2. Compute the asymptotic running time of Merge as function of k, l (you can assume that comparing numbers takes constant time).
3. Compute the asymptotic running time of Sort as a function of n.

Problem 2

Let $A \in Z^{n \times n}$ such that each absolute value of an entry is bounded by U. Show that the coefficients of the characteristic polynomial of A have binary encoding length polynomial in $\log U$ and n.

Problem 3

Consider the following linear program:

$$
\begin{array}{llllllllll}
\max & 8x & + & 4y & + & 2z & & & \\
\text{s.t.} & -x & + & 2y & + & 2z & \leq & 6, & (1)\\
 & 2x & - & 6y & - & z & \leq & 2, & (2)\\
 & 2x & - & 3y & + & 4z & \leq & 1, & (3)\\
 & -x & & & & & \leq & 0, & (4)\\
 & & - & y & & & \leq & 0, & (5)\\
 & & & & - & z & \leq & 0. & (6)
\end{array}
$$

Solve the linear program using the simplex algorithm with *Bland's pivoting rule.* Start with the basis $B = \{4, 5, 6\}$ and the corresponding vertex $(0, 0, 0)^T$.
For each iteration of the simplex algorithm, indicate the current basis and the corresponding vertex (basic feasible solution).

Provide the optimal solution, its value and the certificate of optimality.
Here are the inverse matrices of all the feasible bases.

$$B = \{1,3,4\} \implies A_B^{-1} = \begin{bmatrix} 0 & 0 & -1 \\ 2/7 & -1/7 & -4/7 \\ 3/14 & 1/7 & 1/14 \end{bmatrix} \quad B = \{1,3,6\} \implies A_B^{-1} = \begin{bmatrix} 3 & 2 & 14 \\ 2 & 1 & 8 \\ 0 & 0 & -1 \end{bmatrix}$$

$$B = \{1,4,6\} \implies A_B^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 1/2 & -1/2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad B = \{3,4,5\} \implies A_B^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1/4 & 1/2 & -3/4 \end{bmatrix}$$

$$B = \{3,5,6\} \implies A_B^{-1} = \begin{bmatrix} 1/2 & -3/2 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad B = \{4,5,6\} \implies A_B^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Problem 4

Consider the following linear program:

$$\begin{aligned} \max \quad & -2x_1 - 3x_2 + 4x_3 \\ \text{s.t.} \quad & -2x_1 - 4x_3 \geq -4, \\ & x_1 + 3x_3 - 2x_4 = 3, \\ & x_1, \ldots, x_4 \geq 0. \end{aligned}$$

(a) Reformulate the linear program in the form

$$\max\{\widetilde{c}^T x \colon \widetilde{A}x \leq \widetilde{b},\, x \geq 0\}.$$

for a matrix $\widetilde{A}$ and vectors $\widetilde{b}$, $\widetilde{c}$.

(b) Describe the dual linear program.

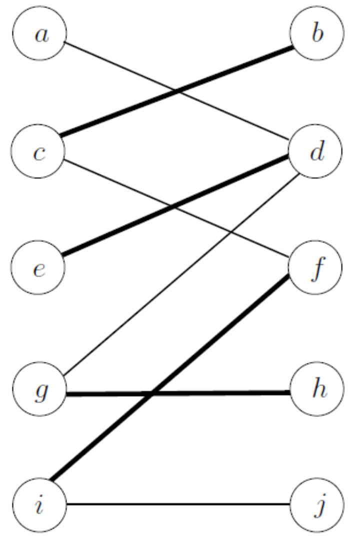
(c) Show that the primal program is feasible and bounded by using duality. Then, give a feasible solution for the primal program.

Problem 5

Let $A \in \mathbb{R}^{m \times n}$ such that A does not have full column rank. We can then find with Gaussian elimination an invertible matrix $U \in \mathbb{R}^{n \times n}$ with $A \cdot U = (A' \; \mathbf{0})$ where $A' \in \mathbb{R}^{m \times n'}$ has full column rank and $\mathbf{0}$ is a $m \times (n - n')$ matrix of zeros. Show that the system $A'x \leq b$ is then feasible if and only if $Ax \leq b$ is feasible. In particular, let x' be a feasible solution of $A'x \leq b$ and suppose that U from above is given. Show how to compute a feasible solution $\tilde{x}$ of $Ax \leq b$. Also vice versa, show how to compute x', if $\tilde{x}$ is given.

Problem 6

1. Consider the following bipartite graph with a matching consisting of the bold black edges. Prove or disprove that this is a maximum cardinality matching.



2. Show that in any graph the size of any maximal matching M (i.e. a matching in which one cannot add an edge while keeping it a matching) is at least half the size of a maximum matching M^*. Provide an example of a graph with a maximal matching whose size is exactly half the size of the maximum matching.