École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand
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December 16, 2016

Combinatorial Optimization (Fall 2016)

Assignment 11

Deadline: December 23 10:00, into the right box in front of MA C1 563.

Exercises marked with a $\star$ can be handed in for bonus points.

Problem 1

Let $A \in \mathbb{R}^{n \times n}$ be an invertible matrix and $b \in \mathbb{R}^n$ a vector. In class we defined the ellipsoid $E(A, b)$ as the imagine of the unit ball under the linear mapping $t(x) = Ax + b$. Show that

$$E(A,b) = \{x \in \mathbb{R}^n : (x-b)^\top A^{-\top} A^{-1}(x-b) \leq 1\}$$

Problem 2

Prove the Hyperplane Separation Theorem: if $K \subset \mathbb{R}^n$ is convex and closed and $x^* \notin K$, then there is an hyperplane $a^\top x = b$ such that $a^\top x^* > b$ and $a^\top x < b$ for any $x \in K$.

Problem 3 ($\star$)

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a full dimensional 0/1 polytope (i.e. the vertices of P have 0/1 coordinates) and $c \in \mathbb{Z}^n$. We will show how we can use the ellipsoid method to solve the optimization problem $\max\{c^\top x : x \in P\}$.

Define $z^* := \max\{c^\top x : x \in P\}$ and $c_{\max} := \max\{|c_i| : 1 \leq i \leq n\}$.

(i) Show that the ellipsoid method requires $O(n^2 \log(n \cdot c_{\max}))$ iterations to decide, for some integer β, whether $P \cap \{c^\top x \geq \beta - \frac{1}{2}\}$ is full dimensional or not. (You only need to find a suitable initial ellipsoid and stopping value L. To find the latter, start from a simplex contained in P and transform it so that it is contained in $P \cap \{c^\top x \geq \beta - \frac{1}{2}\}$)

(ii) Show that we can use binary search to find z^* with $\log(n \cdot c_{\max})$ calls to the ellipsoid method.

(iii) Using part (i), (ii) we can find the optimal value z^* and a point $y \in P \cap \{c^\top x \geq z^* - \frac{1}{2}\}$. Show how you can use this to find an optimal solution x^* such that $c^\top x^* = z^*$ in time polynomial in $n, c_{\max}$ and the number of facets of P.