École Polytechnique Fédérale de Lausanne
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December 16, 2016

Combinatorial Optimization (Fall 2016)

Assignment 11

Deadline: December 23 10:00, into the right box in front of MA C1 563.

Exercises marked with a $\star$ can be handed in for bonus points.

Problem 1
Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix and $b \in \mathbb{R}^n$ a vector. In class we defined the ellipsoid $E(A,b)$ as the imagine of the unit ball under the linear mapping $t(x) = Ax + b$. Show that

$$E(A,b) = \{x \in \mathbb{R}^n : (x-b)^\top A^{-\top} A^{-1}(x-b) \leq 1\}$$

Solution:

Proof sketch. It follows from the fact that

$$||A^{-1}(y-b)|| = (y-b)^\top A^{-\top} A^{-1}(y-b)$$

Problem 2
Prove the Hyperplane Separation Theorem: if $K \subset \mathbb{R}^n$ is convex and closed and $x^* \notin K$, then there is an hyperplane $a^\top x = b$ such that $a^\top x^* > b$ and $a^\top x < b$ for any $x \in K$.

Solution:
Consider the function $f(x) = ||x^* - x||_2$ for $x \in K$. f is continuous and there is $y \in K$ such that $f(y)$ is minimum: to see this, just choose a point $c \in K$ and consider the ball B centered at c of radius $||c - x^*||_2$, then we can restrict f to $K \cap B$, which is bounded and closed, hence compact, so there is a y minimizing $f(y)$ over $K \cap B$. It is immediate that y is a minimum for all K. Now consider the segment x^*y, and let $a^\top x = b$ the hyperplane whose normal is the direction of x^*y and containing its midpoint. Clearly $a^\top x^* > b$. Assume by contradiction that there is any point $y' \in K$ such that $a^\top y' \geq b$. Then by convexity yy' is contained in K and it intersects the ball centered at x^* of radius $f(y)$ in an internal point, contradicting the minimality of $f(y)$.

Problem 3 ($\star$)
Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a full dimensional 0/1 polytope (i.e. the vertices of P have 0/1 coordinates) and $c \in \mathbb{Z}^n$. We will show how we can use the ellipsoid method to solve the optimization problem $\max\{c^\top x : x \in P\}$.

Define $z^* := \max\{c^\top x : x \in P\}$ and $c_{\max} := \max\{|c_i| : 1 \leq i \leq n\}$.

(i) Show that the ellipsoid method requires $O(n^2 \log(n \cdot c_{\max}))$ iterations to decide, for some integer β, whether $P \cap \{c^\top x \geq \beta - \frac{1}{2}\}$ is full dimensional or not. (You only need to find a suitable initial ellipsoid and stopping value L. To find the latter, start from a simplex contained in P and transform it so that it is contained in $P \cap \{c^\top x \geq \beta - \frac{1}{2}\}$)

(ii) Show that we can use binary search to find z^* with $\log(n \cdot c_{\max})$ calls to the ellipsoid method.

(iii) Using part (i), (ii) we can find the optimal value z^* and a point $y \in P \cap \{c^\top x \geq z^* - \frac{1}{2}\}$. Show how you can use this to find an optimal solution x^* such that $c^\top x^* = z^*$ in time polynomial in $n, c_{\max}$ and the number of facets of P.

Solution:

[(i)] First note that all 0/1 polytopes are contained inside the ball B centered at $\frac{1}{2}\vec{1}$ with radius $\sqrt{n}/2$. We can upper bound the volume of this ball by $\sqrt{n}^n$. Let $P' := P \cap \{c^\top x \geq \beta - \frac{1}{2}\}$. We want to lower bound the volume of P', in the case that it is full dimensional. Let x_0 be an integral vertex in P'. We saw in class that since P is full dimensional it contains a simplex $\Delta = \text{conv}\{x_0, x_1, \cdots, x_n\}$ of volume at least $1/n!$. The idea is to scale this simplex so that it is contained in P'. We define $\alpha = \frac{1}{2nc_{\max}}$ and consider the simplex $\Delta' = [z_0, z_1, \cdots, z_{n-1}]$ where $z_i = x_0 + \alpha(x_i - x_0)$. To see that Δ' is contained in P' we can check that each z_i is in P and satisfies $c^\top z_i \geq \beta - \frac{1}{2}$. Then we obtain that

$$vol(P') \geq vol(\Delta') \geq \frac{1}{n!}\left(\frac{1}{2nc_{\max}}\right)^n,$$

where the last inequality follows from the fact that $z_i - z_0 = \alpha(x_i - x_0)$ for $i = 1, \ldots, n$ i.e. the length of each edge of the simplex is scaled by α, so the volume of Δ' is at least the volume of Δ scaled by α^n. Setting $L = \frac{1}{n!}\left(\frac{1}{2nc_{\max}}\right)^n$ and using the fact that the ellipsoid method terminates in $O(n \log(vol(B)/L))$ gives us the correct bound.

[(ii)] We use the fact that the value of $c^\top x$ lies between $-nc_{\max}$ and $nc_{\max}$ for any vertex x of P to conclude that z^* must also lie within these bounds. Using binary search on this interval of integer points takes $\log(nc_{\max})$ steps.

[(iii)] The algorithm described in (i) and (ii) gives us the optimal value z^* but also a point $y \in P' = P \cap \{c^\top x \geq z^* - \frac{1}{2}\}$. Now, any vertex v of P' such that $c^\top v \geq c^\top y$ will achieve the optimal objective value. To find v, one can proceed as follows. We start by projecting y onto the hyperplane $c^\top x = z^*$. Let y' be the projection. If $y' \in P$ then we are done. Otherwise we can find a point on the line segment yy' that intersects a facet F of P (this can be done by first checking all the facets until we find one intersecting yy', and then solving a linear system to find the intersection point). We have now reduced the dimension of our problem by one and can proceed by once again projecting this new point onto the hyperplane $c^\top x = z^*$ (which, intersected with the facet F, is an hyperplane in dimension $n-1$). Continuing in this way we will arrive at an optimal solution in polynomial time.