École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand

May 18, 2018

Discrete Optimization (Spring 2018)

Assignment 11

Problem 4 could be **submitted** until May 25 12:00 noon into the right box in front of MA C1 563.You are allowed to submit your solutions in groups of at most three students.

Problem 1

Let $M \in \mathbb{Z}^{n\times m}$ be totally unimodular. Prove that the following matrices are totally unimodular as well:

1. M^T
2. $(M \;\; I_n)$
3. $(M \;\; -M)$
4. $M \cdot (I_n - 2e_j^T e_j)$ for some j.

I_n is the $n \times n$ identity matrix and e_j is the vector having a 1 in the j-th component, and 0 in the other components.

Solution:

1. Let A be a square submatrix of M^T. Then $\det(A) = \det(A^T) \in \{-1, 0, 1\}$ as A^T is a square submatrix of M and M is totally unimodular.
2. Let A be a square submatrix of $(M \;\; I_n)$.Let $a_1, \dots, a_k$ be the columns of A that originate from I_n. Each of these columns has as most one 1-entry, the other entries are 0. Hence, using Laplace-expansion successively along these columns we get that $|\det(A)| = |\det(A')| \in \{-1, 0, 1\}$ for some square submatrix A' of M.
3. Let A be a square submatrix of $(M \;\; -M)$. Let $a_1, \dots, a_k$ be the columns of A that originate from $-M$. Let A' be the matrix obtained from A by multiplying $a_1, \dots, a_k$ by -1. Hence $|\det(A)| = |\det(A')|$. Now we have that either A' is a square submatrix of M, hence we are done, or A' has two identical columns, hence it has determinant 0.
4. Observe that $M \cdot (I_n - 2e_j^T e_j)$ is obtained from M by multiplying one column by -1. Thus,$M \cdot (I_n - 2e_j^T e_j)$ is (up to permutation of columns) a submatrix of $(M \;\; -M)$, and we are done thanks to part 3.

Problem 2

Let G be a graph and let A be its node-edge incidence matrix. We have seen in class that if G is bipartite then A is totally unimodular. Prove the converse, *i.e.*, if A is totally unimodular then G is bipartite.

Solution:

We prove the contrapositive. Assume G is not bipartite, we show that its node-edge incidence matrix A is not totally unimodular. Recall that a graph is bipartite if and only if there exists no odd cycle in G. Hence, G has an odd cycle. Let C be this cycle and consider the $|C| \times |C|$ submatrix $\tilde{A}$ of A that describes it, *i.e.*, $\tilde{A}$ is the restriction of A to only the nodes and edges of C. Subsequently, we show that $\det(\tilde{A}) = \pm 2$. Note that each permutation of two rows or columns, respectively, changes the determinant by a factor of (-1). Hence, by permuting the columns and rows of $\tilde{A}$, for the sake of our claim we can assume that

$$\tilde{A} = \begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \\ 1 & & & & 1 \end{bmatrix}$$

But this gives

$$\det(\tilde{A}) = (-1)^0 \det\left(\begin{bmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}\right) + (-1)^{|C|-1} \det\left(\begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \end{bmatrix}\right) = 1 + 1 = 2,$$

where we have used the fact that $|C|$ is odd. This implies that A is not totally unimodular.

Problem 3

Consider a bipartite graph $G = (A \cup B, E)$. Assume there exist matchings M_A and M_B covering vertices $A_1 \subseteq A$ and $B_1 \subseteq B$, respectively. Prove that there always exists a matching that covers $A_1 \cup B_1$.
Hint: The symmetric difference $M_A \Delta M_B$ consists of only cycles and paths.
Solution:
We construct a matching M from M_A and M_B that covers $A_1 \cup B_1$. First, consider $M_A \cup M_B$. Clearly, this set covers both A_1 and B_1 but must not necessarily be a matching. Subsequently, we show how to delete edges from $M_A \cup M_B$ such that it becomes a matching but still covers A_1 and B_1.
Moreover, note that $M_A \cap M_B$ is a matching but does not cover $A_1 \cup B_1$. Thus we only need to remove edges from the symmetric difference $M_A \Delta M_A$, in order to extract a matching whilst preserving the property of covering all $A_1 \cup B_1$.

Claim 1 : $M_A \Delta M_B$ can be decomposed into maximal disjoint connected components.
In order to prove the claim, define a equivalence relation on the set $V_{M_A \Delta M_B}$ of all vertices by $x \sim y \Leftrightarrow x$ and y are connected in $M_A \Delta M_B$. The claim follows readily (consider the equivalence classes for this relation).
Note that this claim is true in general for any graph.

Claim 2 : $\forall\, v \in V_{M_A \Delta M_B}$, $\delta(v) = 1$ or 2.
For any vertex v in the symmetric difference, we have a lower bound the degree on v given by $\delta(v) \geq 1$. This is true because all vertices have to be connected with another vertex since they are all part of a matching (either M_A or M_B).
Moreover, the degree $\delta(v)$ is bounded by above by the fact that if v is in the symmetric difference, it can be linked with at most two edges, one in each matching. Therefore $\delta(v) \leq 2$.
This allows us to conclude that the symmetric difference consists only of paths and cycles, as suggested in the hint. Moreover, by Claim 1, it is sufficient to find matching in each of the connected components of $M_A \Delta M_B$. Then the union of the matchings obtained for each component is still a matching by disjointness.

Now we separate cases:

- **Cycles**: as the graph is bipartite, all cycles must have even length. Thus we can delete every second edge: for instance, for a cycle $C \subseteq M_A \Delta M_B$, we could keep only edges of $M_A \cap C$. Note that in this case, all vertices of C are covered and thus we have a matching that covers all cycles of the symmetric difference.

- **Paths of odd length**: we start with either side of the path and we take one edge every two edges, including the first one. The side where we start does not matter since the path is of odd length.

- **Paths of even length**: the graph being bipartite, both endpoints of the path must be in the same set A or B (simply because a path must be alternating between A and B in a bipartite graph). Assume, w.l.o.g, that they are in A. We show that not both endpoints can also be in A_1. Assume that one of them is in A_1. The edges of the path are alternating (between M_A and M_B). Hence, the edge incident to this one endpoint is in M_A. But then, the edge incident to the other endpoint stems from M_B (since the path has even length). Thus, this other endpoint is not in A_1 (recall we are considering the symmetric difference and if this endpoint was in A_1 it would not be an endpoint).

  And thus for every path of even length we can start from its end point which is in $A_1 \cup B_1$ and delete every second edge without problems.

Overall we have created matchings that cover every connected component of the symmetric difference. Considering the union of all these matching with $M_A \cap M_B$ yields a matching that covers $A_1 \cup B_1$.

Problem 4 ($\star$)

Given a graph G, a perfect matching of G is a matching which covers all the vertices (equivalently, a matching of cardinality $|V|/2$). Suppose you are given an oracle that, given a graph G, tells you whether G has a perfect matching or not. Show how to use this oracle to determine the maximum cardinality matching of a graph $G(V, E)$.
Hint: you should modify the graph at each call of the oracle. The total number of calls (to find the cardinality of the maximum matching, and then to find the matching itself) should be at most $|V| + |E|$.

Solution:
For $k = 0, \dots, n = |V|$, let $G + k$ be the graph obtained by adding to G k dummy vertices joint to all the vertices of G. Since a matching has an even number of nodes, in what follows we only consider values of k of the same parity as n. Notice that $G + k$ has a perfect matching if and only if G has a matching of size $\frac{n-k}{2}$. We call the oracle on $G + k$, starting with $k = 0$ (or 1, depending on the parity of n) and increasing it, until $G + k$ has a perfect matching. For the minimum such k, we know that there is a matching M of size $\frac{n-k}{2}$ and it has maximum cardinality. Now to find such a matching, we remove one edge $e \in E$ from $G + k$ at time and we ask the oracle if this graph has a perfect matching: if it doesn't, then $e \in M$, and we can delete e and its endpoints and continue. If it does, then $e \notin M$ and we delete just e and continue. In this way we will find M. The total number of calls is at most $k + |E| \leq n + |E|$.

Problem 5

A family of sets $\mathcal{C} \subset 2^{[n]}$ is a chain if for all $S, T \in \mathcal{C}$ we have either $S \subseteq T$ or $T \subseteq S$. Suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ are two chains. Let $A \in \{0,1\}^{|\mathcal{C}_1|+|\mathcal{C}_2|\times n}$ with $A_{S,i} = 1$ if $i \in S$ and 0 otherwise, for $i = 1, \dots, n$ and $S \in \mathcal{C}_1 \cup \mathcal{C}_2$. Prove that A is totally unimodular. *Hint: use induction on the size of a square submatrix of* A.

Solution:
By performing elementary row operations (that do not modify the absolute value of the determinant

of any square submatrix), since $\mathcal{C}_1$ and $\mathcal{C}_2$ are chains we can assume without loss of generality that every column of A has at most one 1 within the rows in $\mathcal{C}_1$ and at most one 1 within the rows in $\mathcal{C}_2$. Let B be any square submatrix of A of size k, we perform induction on k. The result is immediate for $k = 1$ as A has $0/1$ entries. For $k \geq 2$, there are several cases:

- There is an all zero column in B. Then $\det(B) = 0$.
- Every column of B has two 1's, one in a row in $\mathcal{C}_1$, one in a row of $\mathcal{C}_2$. Then , summing all the rows in $\mathcal{C}_1$ and subtracting all rows in $\mathcal{C}_2$, gives the 0 vector, hence $\det(B) = 0$.
- There is a column of B with exactly one 1. Then we can expand along this column to compute the determinant and the result follows by induction.