Problem 1
Given the weighted graph on the right, find the following:

a) A matching that is not perfect and has weight 15.

b) A \( w \)-vertex cover of weight 16 where at least seven vertices have non zero weights.

Problem 2 (*)
Let \( G \) be a graph and let \( A \) be its node-edge incidence matrix. We have seen in class that if \( G \) is bipartite then \( A \) is totally unimodular. Prove the converse, i.e., if \( A \) is totally unimodular then \( G \) is bipartite.

Problem 3
Consider a bipartite graph \( G = (U \cup W, E) \). Assume there exist matchings \( M_U \) and \( M_W \) covering vertices \( U_1 \subseteq U \) and \( W_1 \subseteq W \), respectively. Prove that there always exists a matching that covers \( U_1 \cup W_1 \).

Hint: The symmetric difference \( M_U \Delta M_W \) consists of only cycles and paths.

Problem 4
Given a graph \( G = (V, E) \), a subset \( S \subseteq V \) is called stable (or independent) if \( |e \cap S| \leq 1 \) for each \( e \in E \). The independent set problem (ISP) is to find a maximum cardinality stable set on \( G \).

We know that an optimal (but not-necessarily integral) solution of a linear program (LP) can be found in polynomial time. Show that the ISP can be solved in polynomial time if \( G \) is bipartite.

a) Let \( \bar{A} \), \( \bar{b} \), \( \bar{c} \) and \( x^* \) be given as the input, where \( x^* \) is an optimal solution of \( \max \{ c^T x : x \in P \} \) and \( P = \{ x \in \mathbb{R}^n : \bar{A}x \leq \bar{b} \} \). If \( x^* \) is not a vertex of \( P \), argue that one can find a vertex \( \bar{x} \) of \( P \) in polynomial time such that \( \bar{c}^T \bar{x} = \bar{c}^T x^* \).

b) Formulate an LP relaxation of the ISP in the form

\[
\max \quad c^T x \\
\text{s.t.} \quad Ax \leq b \\
\quad x \geq 0
\]

such that \( A \in \mathbb{Z}^{m \times n} \) is totally unimodular for bipartite \( G \), \( b \in \mathbb{Z}^m \) and the corresponding polytope is a convex hull of indicator vectors of stable sets on \( G \). Additionally, require that the encoding length of \( A \), \( b \) and \( c \) is polynomial in \( |V| \) and \( |E| \).