École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand
Assistant: Manuel Aprile
December 9, 2016

Combinatorial Optimization (Fall 2016)

Assignment 10

Deadline: December 16 10:00, into the right box in front of MA C1 563.

Exercises marked with a $\star$ can be handed in for bonus points.

Problem 1

Recall that in class we defined a chain $\mathcal{C}$ to be a family of sets such that for all $S, T \in \mathcal{C}$ we have either $S \subseteq T$ or $T \subseteq S$. Suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ are two chains. Let A be the matrix with rows χ^S for all $S \in \mathcal{C}_1 \cup \mathcal{C}_2$. Prove that A is totally unimodular. That is, show that for all square submatrices B of A we have $\det(B) \in \{0, \pm 1\}$.

Problem 2 ($\star$)

Given a graph $G(V, E)$, the spanning tree polytope $P_{ST}(G)$ is defined as follows:

$$\begin{aligned} P_{ST}(G) = \{x \in \mathbb{R}^E : \; & x(E(U)) \leq |U| - 1 \qquad \forall\, U \subset V \\ & x(E) = |V| - 1 \\ & x \geq 0\} \end{aligned}$$

We will show that each vertex of $P_{ST}(G)$ is integral (i.e. $P_{ST}(G)$ is the convex hull of the incidence vectors of the spanning trees of G) by an uncrossing argument, similarly as seen in class. Given a finite set X, two sets $A, B \subset X$ are called intersecting if $A \cap B, A \setminus B, B \setminus A$ are all non-empty; a family $\mathcal{L} \subset 2^X$ is said to be laminar if no two sets $A, B \in \mathcal{L}$ are intersecting.
Given x^* a vertex of $P_{ST}(G)$, let $\mathcal{F} = \{U \subset V : x^*(E(U)) = |U| - 1\}$.

1. Let $A, B \in \mathcal{F}$, show that $A \cap B, A \cup B \in \mathcal{F}$.
2. Show that if $\mathcal{L}$ is a maximal laminar subfamily of $\mathcal{F}$, then $span(\mathcal{L}) = span(\mathcal{F})$ (where $span(\mathcal{F}) = span\{\chi^{E(A)}, A \in \mathcal{F}\}$, and similarly for $\mathcal{L}$).
3. Let L be the matrix with rows χ^S for $S \in \mathcal{L}$, where $\mathcal{L}$ is laminar. Use the fact that L is totally unimodular (which can be proven similarly as in Problem 1) to show that x^* has integer coordinates only.