École Polytechnique Fédérale de Lausanne
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Combinatorial Optimization (Fall 2016)

Assignment 10

Deadline: December 16 10:00, into the right box in front of MA C1 563.

Exercises marked with a $\star$ can be handed in for bonus points.

Problem 1

Recall that in class we defined a chain $\mathcal{C}$ to be a family of sets such that for all $S, T \in \mathcal{C}$ we have either $S \subseteq T$ or $T \subseteq S$. Suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ are two chains. Let A be the matrix with rows χ^S for all $S \in \mathcal{C}_1 \cup \mathcal{C}_2$. Prove that A is totally unimodular. That is, show that for all square submatrices B of A we have $\det(B) \in \{0, \pm 1\}$.

Solution:
As seen in class, since $\mathcal{C}_1$ and $\mathcal{C}_2$ are chains we can assume without loss of generality that every column of A has at most one 1 within the rows in $\mathcal{C}_1$ and at most one 1 within the rows in $\mathcal{C}_2$ since if this was not the case we can just perform elementary row operations without modifying the determinant of any square submatrix. Let B be any square submatrix of A of size k, we perform induction on k. The result is immediate for $k = 1$ as A has 0/1 entries. For $k \geq 2$, there are several cases:

- There is an all zero column in B. Then $\det(B) = 0$.
- Every column of B has two 1's, one in a row in $\mathcal{C}_1$, one in a row of $\mathcal{C}_2$. Then , summing all the rows in $\mathcal{C}_1$ and subtracting all rows in $\mathcal{C}_2$, gives the 0 vector, hence $\det(B) = 0$.
- There is a column of B with exactly one 1. Then we can expand along this column to compute the determinant and the result follows by induction.

Problem 2 ($\star$)

Given a graph $G(V, E)$, the spanning tree polytope $P_{ST}(G)$ is defined as follows:

$$\begin{aligned} P_{ST}(G) = \{x \in \mathbb{R}^E : x(E(U)) &\leq |U| - 1 \qquad\qquad \forall\, U \subset V \\ x(E) &= |V| - 1 \\ x &\geq 0\} \end{aligned}$$

We will show that each vertex of $P_{ST}(G)$ is integral (i.e. $P_{ST}(G)$ is the convex hull of the incidence vectors of the spanning trees of G) by an uncrossing argument, similarly as seen in class. Given a finite set X, two sets $A, B \subset X$ are called intersecting if $A \cap B, A \setminus B, B \setminus A$ are all non-empty; a family $\mathcal{L} \subset 2^X$ is said to be laminar if no two sets $A, B \in \mathcal{L}$ are intersecting.
Given x^* a vertex of $P_{ST}(G)$, let $\mathcal{F} = \{U \subset V : x^*(E(U)) = |U| - 1\}$.

1. Let $A, B \in \mathcal{F}$, show that $A \cap B, A \cup B \in \mathcal{F}$.
2. Show that if $\mathcal{L}$ is a maximal laminar subfamily of $\mathcal{F}$, then $span(\mathcal{L}) = span(\mathcal{F})$ (where $span(\mathcal{F}) = span\{\chi^{E(A)}, A \in \mathcal{F}\}$, and similarly for $\mathcal{L}$).

3. Let L be the matrix with rows χ^S for $S \in \mathcal{L}$, where $\mathcal{L}$ is laminar. Use the fact that L is totally unimodular (which can be proven similarly as in Problem 1) to show that x^* has integer coordinates only.

Solution:

1. We have:

$$|A| - 1 + |B| - 1 = x^*(E(A)) + x^*(E(B)) \leq x^*(E(A \cup B)) + x^*(E(A \cap B))$$

where the inequality follows since the edges in $E(A \cap B)$ are counted twice and each other edge induced by A or B is also induced by $A \cup B$. Now,

$$x^*(E(A \cup B)) + x^*(E(A \cap B)) \leq |A \cup B| - 1 + |A \cap B| - 1 = |A| - 1 + |B| - 1$$

hence all the inequalities hold with equality and in particular $x^*(E(A \cup B)) = |A \cup B| - 1$ and $x^*(E(A \cap B)) = |A \cap B| - 1$.

2. Similarly as in the proof seen in class, for $A \in \mathcal{F}$ we define $viol(A) = \{B \in \mathcal{L} : A, B$ are intersecting$\}$. Assume by contradiction that $span(\mathcal{L})$ is a strict subset of $span(\mathcal{F})$, and let A such that $\chi^A \in span(\mathcal{F}) \setminus span(\mathcal{L})$ and $|viol(A)|$ is minimum. By maximality of $\mathcal{L}$, $|viol(A)| \geq 1$ otherwise $\mathcal{L} \cup A$ would be a larger laminar family contained in $\mathcal{F}$. Hence let $B \in viol(A)$, we claim that $|viol(A \cap B)| < |viol(A)|$. Indeed, let $C \in viol(A \cap B)$, $C \neq B$, we have that $C \setminus A \cap B, A \cap B \setminus C, A \cap B \cap C$ are non-empty. Moreover, $C \in \mathcal{L}$, hence either $C \subset B$, or $B \subset C$ or $B \cap C = \emptyset$. The last one is not possible as $A \cap B \cap C \subset B \cap C$ Assume $C \subset B$: then if $C \subset A$, $C \subset A \cap B$, a contradiction to $C \setminus A \cap B$ being non-empty. If $A \subset C$, then $A \subset B$, a contradiction to $B \in viol(A)$. If $A \cap C = \emptyset$, then we get again a contradiction to $A \cap B \cap C$ being non-empty. Hence in this case A, C are intersecting and the claim is proved. In the case $B \subset C$, the claim is proved similarly. With analogous arguments one proves that $|viol(A \cup B)| < |viol(A)|$. Now, by minimality of $|viol(A)|$, we must have that $\chi^{E(A \cup B)}, \chi^{E(A \cap B)} \in span(\mathcal{L})$, but then $\chi^{E(A)} = \chi^{E(A \cup B)} + \chi^{E(A \cap B)} - \chi^{E(B)} \in span(\mathcal{L})$, a contradiction. (Notice that the equality holds because $A, B \in \mathcal{F}$ as seen in the proof of part 1).

3. x^* is the unique solution of the system $x(E(U)) = |U| - 1$ for

$$\begin{aligned} x(E(U)) &= |U| - 1 \;\; \forall\, U \in \mathcal{F} \\ x_e &= 0 \;\; \forall\, e \in \bar{E} \end{aligned}$$

for some $\bar{E} \subset E$. Using part 2, we can reduce the system to

$$\begin{aligned} x(E(U)) &= |U| - 1 \;\; \forall\, U \in \mathcal{L} \\ x_e &= 0 \;\; \forall\, e \in \bar{E} \end{aligned}$$

Now, the matrix associated to the system is totally unimodular, hence x^* is an integer vector.