École Polytechnique Fédérale de Lausanne
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December 2, 2016

Combinatorial Optimization (Fall 2016)

Assignment 9

Deadline: December 9 10:00, into the right box in front of MA C1 563.

Exercises marked with a $\star$ can be handed in for bonus points.

Problem 1

Give an example of two matroids $M_1(E, \mathcal{I}_1), M_2(E, \mathcal{I}_2)$ such that $(E, \mathcal{I}_1 \cap \mathcal{I}_2)$ is not a matroid.

Solution:
There are many easy examples. For instance, consider the matroids given in class whose intersection is the set of matchings of a bipartite graph. One can easily see that the set of matchings is not a matroid by giving an example of a graph with two maximal matchings of different cardinality (which violates the third matroid axiom).

Problem 2

Assume E is a finite set and $r : 2^E \to \mathbb{N}$ is a function that satisfies:

(i) $r(A) \leq |A|$ for any $A \subseteq E$.

(ii) If $A \subseteq B \subseteq E$, $r(A) \leq r(B)$.

(iii) r is submodular.

Show that $M(E, \mathcal{I})$ is a matroid, where $\mathcal{I} = \{A \subseteq E : r(A) = |A|\}$.

Solution:
We show that the three matroid axioms hold for $M(E, \mathcal{I})$.

1. $\emptyset \in \mathcal{I}$: indeed, $r(\emptyset) \leq |\emptyset| = 0$ hence $r(\emptyset) = 0$.

2. Let $A \subset B$, and $B \in \mathcal{I}$, i.e. $r(B) = |B|$. By (i), (iii) we have

$$|B| = r(B) \leq r(A) + r(B \setminus A) \leq |A| + |B \setminus A| = |B|$$

   which implies the inequalities hold with equality. Now if $r(A) < |A|$, then $r(B \setminus A) > |B \setminus A|$, contradicting (i).

3. Let $A, B \in \mathcal{I}$ with $|A| < |B|$, we show that there exists $e \in B \setminus A$ such that $A + e = A \cup \{e\} \in \mathcal{I}$. First assume $B \setminus A = \{e\}$, then $|A| < |B|$ implies $B = A + e$ and we are done.

   Suppose now $B \setminus A = \{e_1, \ldots, e_k\}$, $k \geq 2$. Assume that $A + e_i \notin \mathcal{I}$ for $i = 1, \ldots, k$, but then $r(A) \leq r(A + e_i) < |A| + 1$, hence $r(A + e_i) = r(A)$ for any i. Applying submodularity to $A + e_1, A + e_2$, we get:

$$r(A + e_1) + r(A + e_2) \geq r(A + e_1 + e_2) + r(A)$$

   which implies

$$r(A + e_1 + e_2) \leq r(A) \implies r(A + e_1 + e_2) = r(A) = |A|.$$

If $k = 2$, $B \subseteq A + e_1 + e_2$, which is a contradiction with ii) since $r(B) = |B| > |A| = r(A + e_1 + e_2)$. If $k \geq 3$, applying submodularity to $A + e_1 + e_2, A + e_3$ we get similarly as before that

$$r(A + e_1 + e_2) + r(A + e_3) \geq r(A + e_1 + e_2 + a_3) + r(A) \implies r(A + e_1 + e_2 + e_3) = r(A).$$

If $k = 3$, we again have a contradiction, otherwise we can repeat the argument and eventually reach a contradiction.

Problem 3 ($\star$)

Given an undirected graph $G(V, E)$ and $s, t \in V$, consider the problem of deciding whether there is an $s - t$ path that contains all the vertices. Show that this problem can be formulated as the intersection of three matroids (i.e. give three matroids such that such $s - t$ path exists if and only if there exists a set of a certain cardinality which is independent in all three matroids).

Solution:
Let $D = (V, A)$ be the directed graph obtained from G by replacing each edge uv of G by two arcs (u, v) and (v, u). We then let $M_1 = (A, \mathcal{I}_1)$ be the forest matroid of D, and $M_2 = (A, \mathcal{I}_2)$ and $M_3 = (A, \mathcal{I}_3)$ be partition matroids defined as

$$\mathcal{I}_2 = \{F \subseteq S : |F \cap \delta^-(v)| \leq 1, \forall\, v \neq s,\ |F \cap \delta^-(s)| = 0\}$$
$$\mathcal{I}_3 = \{F \subseteq S : |F \cap \delta^+(v)| \leq 1, \forall\, v \neq t,\ |F \cap \delta^+(t)| = 0\}$$

Now, we claim that there is an $s - t$ path in G containing all vertices (such a path is called an Hamiltonian path) if and only if there exists a set $I \in \mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_2$ such that $|I| = n - 1$, if G has n vertices. Note that such a set would be of maximum cardinality (in particular for M_1) hence if the claim is true, then we can solve the Hamiltonian path problem by finding the largest set in the intersection of three matroids. The first direction is clear: such a path P would have exactly $n - 1$ edges and directing them from s to t gives the desired independent set. For the other direction, let F be the set of edges corresponding to the arcs of I. F is a forest of G and has size $n - 1$ (no two arcs in I can correspond to the same edge, otherwise they would form a cycle), hence it is a spanning tree of G. But since I is independent in $\mathcal{I}_2, \mathcal{I}_3$ as well, every vertex of G has degree at most two in F, and s, t have degree 1. This implies that F is an $s - t$ path, and since it is a spanning tree it contains all the vertices of G.