École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand

April 19, 2018

Discrete Optimization (Spring 2018)

Assignment 8

Problem 1 can be **submitted** until April 27, 12:00 (noon) into the box in front of MA C1 563. You are allowed to submit your solutions in groups of at most three students.

Problem 1 ($\star$)

Consider the following problem. We are given $B \in \mathbb{N}$, and a set of integer points

$$S = \{p \in \mathbb{Z}^n : \ 0 \leq p_i \leq B, \ \forall i = 1, \ldots, n\},$$

whose points are all colored blue but one, which is red. We have an oracle that, given $i \in \{1, \ldots, n\}$ and $\alpha \in \{0, \ldots, B\}$, tells us whether there exists a red point $x^* \in S$ with $x_i^* \leq \alpha$. Give an algorithm to find the red point using $O(n \log(B))$ many oracle calls.

Solution:
We split the problem in n subproblems: for $i \in \{1, \ldots, n\}$, we want to obtain the i-th component of the red point. This is a binary search problem, and we illustrate how to solve it for $i = 1$. We first call the oracle with $\alpha = \lfloor B/2 \rfloor$. If the answer is positive, we call the oracle with $\alpha = \lfloor B/4 \rfloor$. If it is negative, we know that the point is not in the interval $[0, \lfloor B/2 \rfloor]$, so we call the oracle with $\alpha = \lfloor 3B/4 \rfloor$. We continue the process in the same manner. In this way, we are guaranteed to find each component of the red point with $O(\log(B))$ oracle calls, for a total of $O(n \log(B))$ many calls.

Problem 2

Let $P := \{x \in \mathbb{R}^n : \ Ax = b, \, x \geq 0\}$ be a polyhedron and $\min\{cx : x \in P\}$ be the corresponding primal linear program. Assume that all the coefficients of A, b and c are integral and bounded in absolute value by given $B \in N$, and furthermore let $L := B^n n^{n/2}$.

(a) Show the following: If x_1, x_2 are vertices of P and $cx_1 \neq cx_2$, then $|cx_1 - cx_2| \geq 1/L^2$.

(b) Let x^* and y^* be feasible solutions of the primal and dual linear program respectively. Conclude the following from the above: If $|cx^* - by^*| < 1/L^2$, then each vertex x of P with $cx \leq cx^*$ is an optimal solution of the primal.

Solution:
a) Let B_1 and B_2 be the sets of row indices corresponding to basic feasible solutions x_1, x_2, then $x_1 = \begin{bmatrix} A \\ I \end{bmatrix}_{B_1}^{-1} \begin{bmatrix} b \\ 0 \end{bmatrix}_{B_1}$ and $x_2 = \begin{bmatrix} A \\ I \end{bmatrix}_{B_2}^{-1} \begin{bmatrix} b \\ 0 \end{bmatrix}_{B_2}$. A basic result from linear algebra states that given an $n \times n$ invertible matrix M one has that $M^{-1} = \frac{1}{\det(M)} \text{adj}(M)$. If one has that the entries of M are integral and bounded in absolute value by B then Hadamard's bound gives that $\det(M) \leq B^n n^{n/2} = L$. Combining the two statements gives that the entries of M^{-1} are integer multiples of $\frac{1}{|\det(M)|} \geq \frac{1}{L}$. Applying this statement to $\begin{bmatrix} A \\ I \end{bmatrix}_{B_1}$ and since b is integral we obtain that each entry of x_1 is an integer multiple of $1/\delta_1 = 1/|\det(\begin{bmatrix} A \\ I \end{bmatrix}_{B_1})| \geq 1/L$. Analogously we define $1/\delta_2$ for x_2. Thus we obtain that $|cx_1 - cx_2| = |c(x_1 - x_2)|$ is a positive integer multiple of $\frac{1}{\delta_1 \delta_2} \geq 1/L^2$ since $c \in \mathbb{Z}^n$.

b) We prove the statement by contradiction. Assume that x is not an optimal solution, then there is a vertex $\bar{x}$ such that $c\bar{x} < cx$. Then by week duality one has $by^* \leq c\bar{x} < cx \leq cx^*$ which further implies $|c\bar{x} - cx| < 1/L^2$ contradicting a).

Problem 3

Let $Ax \leq b$ be a system of inequalities where each component of A and b is an integer bounded by B in absolute value. Show that $Ax \leq b$ is feasible if and only if $Ax \leq b$, $-B^n \cdot n^{n/2} \cdot n \cdot B \leq x_i \leq B^n \cdot n^{n/2} \cdot n \cdot B, \ \forall i \in [n]$ is feasible.

Hint: Consider a feasible point x^* and the index sets $I = \{i : \ x_i^* \geq 0\}$ and $J = \{j : x_j^* \leq 0\}$. The polyhedron defined by $Ax \leq b, \ x_i \geq 0, \ i \in I, \ x_j \leq 0, \ j \in J$ is feasible and has vertices. Estimate the infinity norm of a vertex.

Solution:

Proof sketch. By using the hint one obtains a system of inequalities with matrix $\bar{A}$ of the full column rank. Thus the corresponding polyhedron has vertices which further implies that there is a subsystem $A'x \leq b'$ such that $x^* = A'^{-1}b'$ and we could use the Hadamarad rule to bound the size of entries of x^*.

Full proof. By using the hint, we obtain a system of inequalities with matrix $\bar{A}$ of the full column rank and vector $\bar{b}$. Indeed, one can see that in the new system $\bar{A}x \leq \bar{b}$, $\bar{A}$ is of the form:

$$\bar{A} = \begin{pmatrix} A \\ X \\ Y \end{pmatrix}$$

and $\bar{b} = \begin{pmatrix} b & 0 \end{pmatrix}^T$, where X is an $|I| \times n$ matrix, the x_{kl} coefficient is equal to 1 if $k = l$ and $k \in I$, Y is an $|J| \times n$ matrix where the y_{kl} coefficient is equal to -1 if $k = l$ and $k \in J$ and 0 otherwise. By construction of $\bar{A}$, we can see that it has full column rank.

This implies that the corresponding polyhedron has vertices. Moreover, by a theorem from the course, a vertex is characterized by a subsystem $A'x \leq b'$ where $\text{rank}(A') = n$ such that $x^\star = A'^{-1}b'$. Hence,

$$|x_i^\star| = |\sum_{j=1}^{n} A'^{-1}_{ij} b_j| \leq B \sum_{j=1}^{n} |A'^{-1}_{ij}|$$

It remains to bound entries of the inverse matrix A'^{-1}. As in Problem 2, we use that $A'^{-1} = \frac{\text{adj}(A')}{\det(A')}$. Since A' is a submatrix of $\bar{A}$, we know that its entries are integers, so $|\det(A')| \geq 1$. From the Hadamard's bound, we have that $|A'^{-1}_{ij}| \leq B^{n-1}(n-1)^{\frac{n-1}{2}}$. By combining those two facts, we have the desired result

$$|x_i^\star| \leq B \sum_{j=1}^{n} |A'^{-1}_{ij}| \leq B \sum_{j=1}^{n} B^{n-1}(n-1)^{\frac{n-1}{2}} \leq n \cdot B \cdot (n-1)^{\frac{n-1}{2}} \cdot B^{n-1} \leq n \cdot B \cdot n^{\frac{n}{2}} \cdot B^n.$$

Problem 4

Let $A \in \mathbb{R}^{n \times n}$ be an invertible matrix and $b \in \mathbb{R}^n$ a vector. The ellipsoid $E(A, b)$ is defined as the image of the unit ball under the linear mapping $t(x) = Ax + b$. Show that

$$E(A, b) = \{x \in \mathbb{R}^n : (x - b)^\top A^{-\top} A^{-1}(x - b) \leq 1\}.$$

Solution:

The ellipsoid $E(A, b)$ is the image of the unit ball by a linear mapping $t(x) = Ax + b$. The unit ball is denoted by $B(0, 1) := \{x \in \mathbb{R}^2 \mid \|x\|_2^2 \leq 1\}$. Hence,

$$
\begin{aligned}
E(A,b) &= \{t(x) \in \mathbb{R}^2 \mid \|x\|_2^2 \leq 1\} \\
&= \{Ax + b \in \mathbb{R}^2 \mid \|x\|_2^2 \leq 1\} \\
&= \{y \in \mathbb{R}^2 \mid \|A^{-1}(y-b)\|_2^2 \leq 1\} \\
&= \{y \in \mathbb{R}^2 \mid (y-b)^\top A^{-\top} A^{-1}(y-b) \leq 1\}.
\end{aligned}
$$

Problem 5

Draw $E(A,b)$ for $A = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Solution:

Using the previous exercise, we have that $E(A,b) = \{y \in \mathbb{R}^2 \mid (y-b)^\top A^{-\top} A^{-1}(y-b) \leq 1\}$. Observe that the matrix $A^{-T}A^{-1}$ is symmetric and it has real entries. The spectral theorem gives that $A^{-T}A^{-1}$ is diagonalizable by an orthogonal matrix S, i.e.,

$$
A^{-T}A^{-1} = SDS^{-1} = SDS^T \approx \begin{pmatrix} -0.86 & 0.51 \\ 0.51 & 0.86 \end{pmatrix} \begin{pmatrix} 38.97 & 0 \\ 0 & 0.026 \end{pmatrix} \begin{pmatrix} -0.86 & 0.51 \\ 0.51 & 0.86 \end{pmatrix},
$$

and therefore

$$
\begin{aligned}
E(A,b) &= \{y \in \mathbb{R}^2 \mid (y-b)^\top SDS^\top (y-b) \leq 1\} \\
&= \{y \in \mathbb{R}^2 \mid (S^\top (y-b))^\top D (S^\top (y-b)) \leq 1\} \\
&= \left\{ y = b + \underbrace{S^{-\top}}_{S} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid x \in \mathbb{R}^2,\ 38.97\ x_1^2 + 0.026\ x_2^2 \leq 1 \right\}.
\end{aligned}
$$

Geometrically, the ellipsoid $E(A,b)$ lives in the orthonormal coordinate system in $\mathbb{R}^2$ centered at b whose principal axes are defined by the eigenvectors $v_1 \approx \begin{pmatrix} -0.86 \\ 0.51 \end{pmatrix}$ and $v_2 \approx \begin{pmatrix} 0.51 \\ 0.86 \end{pmatrix}$ of the matrix $A^{-\top}A^{-1}$, i.e., by the columns of S. Furthermore, lengths of the principal directions of $E(A,b)$ are determined by the eigenvalues $\lambda_1 \approx 38.97$ and $\lambda_2 \approx 0.026$. Namely, those lengths are $1/\sqrt{\lambda_1}$ and $1/\sqrt{\lambda_2}$.

Problem 6

Show that the unit simplex $\Delta = \text{conv}\{0, e_1, \cdots, e_n\} \subset \mathbb{R}^n$ has volume $\frac{1}{n!}$.

Solution:

We will solve the problem using induction on n. Starting with $n = 2$, we have that $\Delta = \Delta((0,0),(1,0),(0,1)) = \text{conv}\{0, e_1, e_2\}$, where $\Delta((0,0),(1,0),(0,1))$ means the triangle between the three vertices $(0,0),(1,0)$ and $(0,1)$. It is clear that the volume of Δ is equal to $\frac{1}{2}$.

For the inductive step we use

$$
\begin{aligned}
\text{vol}_n(\Delta_n) &= \int_0^1 \text{vol}_{n-1}(\alpha \cdot \Delta_{n-1})\, \mathrm{d}\alpha \\
&= \int_0^1 \alpha^{n-1} \text{vol}_{n-1}(\Delta_{n-1})\, \mathrm{d}\alpha \\
&= \frac{1}{(n-1)!} \int_0^1 \alpha^{n-1}\, \mathrm{d}\alpha \\
&= \frac{1}{n!}
\end{aligned}
$$

In this equation, we have use the result that, $\text{vol}_n(\lambda K) = \lambda^n \text{vol}_n(K)$, for any $K \subset \mathbb{R}^n$. We also have used the induction hypothesis in the third equation. Therefore, we get the desired result.