Problem 1 (⋆)
Consider the following problem. We are given $B \in \mathbb{N}$, and a set of integer points
\[ S = \{ p \in \mathbb{Z}^n : 0 \leq p_i \leq B, \forall i = 1, \ldots, n \}, \]
whose points are all colored blue but one, which is red. We have an oracle that, given $i \in \{1, \ldots, n\}$ and $\alpha \in \{0, \ldots, B\}$, tells us whether there exists a red point $x^* \in S$ with $x^*_i \leq \alpha$. Give an algorithm to find the red point using $O(n \log(B))$ many oracle calls.

Problem 2
Let $P := \{ x \in \mathbb{R}^n : Ax = b, x \geq 0 \}$ be a polyhedron and $\min\{cx : x \in P\}$ be the corresponding primal linear program. Assume that all the coefficients of $A$, $b$ and $c$ are integral and bounded in absolute value by given $B \in \mathbb{N}$, and furthermore let $L := B^n n^{n/2}$.

(a) Show the following: If $x_1, x_2$ are vertices of $P$ and $cx_1 \neq cx_2$, then $|cx_1 - cx_2| \geq 1/L^2$.

(b) Let $x^*$ and $y^*$ be feasible solutions of the primal and dual linear program respectively. Conclude the following from the above: If $|cx^* - by^*| < 1/L^2$, then each vertex $x$ of $P$ with $cx \leq cx^*$ is an optimal solution of the primal.

Problem 3
Let $Ax \leq b$ be a system of inequalities where each component of $A$ and $b$ is an integer bounded by $B$ in absolute value. Show that $Ax \leq b$ is feasible if and only if $Ax \leq b + B^n \cdot n^{n/2} \cdot n \cdot B$, $\forall i \in [n]$ is feasible.

Hint: Consider a feasible point $x^*$ and the index sets $I = \{ i : x^*_i \geq 0 \}$ and $J = \{ j : x^*_j \leq 0 \}$. The polyhedron defined by $Ax \leq b$, $x_i \geq 0$, $i \in I$, $x_j \leq 0$, $j \in J$ is feasible and has vertices. Estimate the infinity norm of a vertex.

Problem 4
Let $A \in \mathbb{R}^{n \times n}$ be an invertible matrix and $b \in \mathbb{R}^n$ a vector. The ellipsoid $E(A, b)$ is defined as the image of the unit ball under the linear mapping $t(x) = Ax + b$. Show that
\[ E(A, b) = \{ x \in \mathbb{R}^n : (x - b)^\top A^{-\top} A^{-1}(x - b) \leq 1 \}. \]

Problem 5
Draw $E(A, b)$ for $A = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Problem 6
Show that the unit simplex $\Delta = \text{conv}\{0, e_1, \ldots, e_n\} \subset \mathbb{R}^n$ has volume $\frac{1}{n!}$. 