École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand
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Combinatorial Optimization (Fall 2016)

Assignment 7

Deadline: November 18 10:00, into the right box in front of MA C1 563.

Exercises marked with a $\star$ can be handed in for bonus points.

Problem 1

Given a matroid $M = (E, \mathcal{I})$ and its corresponding matroid polytope P_M, show that its associated system of inequalities $\{x(S) \leq \text{rk}(S) \; \forall S \subset E, x \geq 0\}$ is Totally Dual Integral.

Solution:
We have to show that for any $c \in \mathbb{Z}^m$, the dual of the LP $\max\{cx : x(S) \leq \text{rk}(S) \; \forall S \subset E, x \geq 0\}$ has an optimal integral solution. But this has been shown in the lecture: setting $y_e = c_i - c_{i+1}$ for any e picked by the greedy algorithm, $y_e = 0$ for any other, gives an optimal dual solution that is trivially integral.

Problem 2

Let G be a disconnected graph with connected components G_1, G_2, and let P be the matching polytope of G, and P_i the matching polytope of G_i, $i = 1, 2$. Show that $P = P_1 \times P_2$ (where, given polytopes $A \subset \mathbb{R}^n, B \subset \mathbb{R}^m$, we define their cartesian product $A \times B = \{(x, y) \in \mathbb{R}^{n+m} : x \in A, y \in B\}$).

Solution:
Let us denote by E_i the set of edges of G_i, $i = 1, 2$. Notice that $M \subset E$ a matching in G if and only if $M = M_1 \cup M_2$, where $M_i \subseteq E_i$ is a matching in G_i for $i = 1, 2$. This immediately implies that P and $P_1 \times P_2$ have the same vertices, hence they are the same polytope.

Problem 3

Let $\mathcal{C}$ be the cone generated by linearly independent vectors $a_1, \ldots, a_n \in \mathbb{R}^n$, i.e.

$$\mathcal{C} = \{\sum_{i=1}^{n} \lambda_i a_i : \lambda_i \geq 0 \; \forall i = 1, \ldots, n\}.$$

Show that for any $i = 1, \ldots, n$ there is a point $c \in \mathcal{C} \cap \mathbb{Z}^n$ such that $c + e_i \in \mathcal{C}$.

Solution:
We show that there is a point $c \in \mathcal{C} \cap \mathbb{Z}^n$ in the cone such that a ball of radius 1 and center c is entirely contained in $\mathcal{C}$. To see this, we first show that $\mathcal{C}$ contains arbitrary large balls. Consider the polytope $Q =$conv$\{\pm a_i\}$, which has dimension n since the a_i's are linearly independent. This means that there exist a ball of radius $r > 0$ contained in Q. Clearly, the translate $Q + \sum_i a_i$ is contained in $\mathcal{C}$, and it still contains such ball. By multiplying by an arbitrary large factor k, we have that $k(Q + \sum_i a_i)$ is contained in $\mathcal{C}$ and contains an arbitrary large ball $\mathcal{B}$. Now, it is easy to see that any ball of radius at least $\sqrt{n}/2$ contains an integer point: for instance, it contains a cube with side length 1, which can be written as $v + x : 0 \leq x \leq 1$ for some translation vector v. Then choosing $x = \lfloor v \rfloor - v$ gives us the desired point. By requiring that $\mathcal{B}$ has radius at least $\sqrt{n}/2 + 1$,

we are then sure that the integer point is also surrounded by a ball of radius 1 contained in $\mathcal{B}$, hence in $\mathcal{C}$, and we are done.

Problem 4 ($\star$)

Two vertices x^1, x^2 of a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ are said to be *adjacent* if there is a subsystem $A'x \leq b'$ of $Ax \leq b$ such that $A' \in \mathbb{R}^{(n-1)\times n}$, the rows of A' are linearly independent, and x^1, x^2 satisfy $A'x \leq b'$ with equality.

Let $M = (E, \mathcal{I})$ be a matroid and P_M the corresponding matroid polytope. Given $I_1, I_2 \in \mathcal{I}$ with $I_1 \neq I_2$, show that χ^{I_1} and χ^{I_2} are adjacent vertices of P_M if and only if one of the following conditions hold:

(i) $I_1 \subseteq I_2$ and $|I_1| + 1 = |I_2|$

(ii) $I_2 \subseteq I_1$ and $|I_2| + 1 = |I_1|$

(iii) $|I_1 \setminus I_2| = |I_2 \setminus I_1| = 1$ and $I_1 \cup I_2 \notin \mathcal{I}$

Solution:

Let A, b be such that $P_M = \{x \in \mathbb{R}^E : Ax \leq b\}$, and let $|E| = n$.

($\Leftarrow$) Suppose I_1 and I_2 satisfy (i). Then I_1 and I_2 satisfy:

$$x(e) = rk(e) \qquad \forall e \in I_1 \tag{1}$$
$$x(e) = 0 \qquad \forall e \notin I_2, \tag{2}$$

which corresponds to a submatrix of A with $n-1$ linearly independent rows. The case (ii) is analogous. For the case (iii), we need a slightly different system:

$$x(e) = rk(e) \qquad \forall e \in I_1 \cap I_2 \tag{3}$$
$$x(e) = 0 \qquad \forall e \notin I_1 \cup I_2 \tag{4}$$
$$x(I_1 \cup I_2) = rk(I_1 \cup I_2). \tag{5}$$

It is immediate to see that rows of the corresponding submatrix of A are linearly independent.

($\Rightarrow$) First consider the case where I_1 is not a base of $I_1 \cup I_2$. Then by the third axiom there exists some $j \in I_2 \setminus I_1$ such that $I_1 \cup j \in \mathcal{I}$. Hence

$$\frac{1}{2}\left(\chi^{I_1} + \chi^{I_2}\right) = \frac{1}{2}\left(\chi^{I_1 \cup j} + \chi^{I_2 \setminus j}\right). \tag{6}$$

We now show that this implies that $I_1 \cup j = I_2$ and $I_2 \setminus j = I_1$, meaning that we are in case (i). Indeed, since χ^{I_1} and χI_2 are adjacent, there is a supporting hyperplane that intersects P_M exactly in the segment between χ^{I_1}, χ^{I_2}. Consider any point in the segment (in particular the midpoint): since P_M is a polytope, it is a convex combination of vertices of P_M, but it can only be expressed as combination of χ^{I_1}, χ^{I_2} (all other vertices are on the same side of the supporting hyperplane). Since we expressed this midpoint as the midpoint of another segment between $\chi^{I_1 \cup j}$ and $\chi^{I_2 \setminus j}$ it must be that these are actually the same vertices as χ^{I_1} and χ^{I_2}, and we are done.

The case where I_2 is not a base of $I_1 \cup I_2$ is similar. Hence it remains to consider the case where both I_1 and I_2 are bases of $I_1 \cup I_2$ (hence $I_1 \cup I_2 \notin \mathcal{I}$). Using the strong basis exchange property, there are $i \in I_1 \setminus I_2$, $j \in I_2 \setminus I_1$ such that $I_1 \setminus i \cup j$ and $I_2 \setminus j \cup i$ are both bases of $I_1 \cup I_2$. But then

$$\frac{1}{2}\left(\chi^{I_1} + \chi^{I_2}\right) = \frac{1}{2}\left(\chi^{I_1 \setminus i \cup j} + \chi^{I_2 \setminus j \cup i}\right) \tag{7}$$

and using the same argument as before we have that $I_1 \setminus i \cup j = I_2$ and $I_2 \setminus j \cup i = I_1$, meaning that I_1 and I_2 satisfy (iii).