Exercises marked with a ⋆ can be handed in for bonus points.

Problem 1
Given a matroid $M = (E, \mathcal{I})$ and its corresponding matroid polytope $P_M$, show that its associated system of inequalities $\{x(S) \leq \text{rk}(S) \forall S \subseteq E, x \geq 0\}$ is Totally Dual Integral.

Solution:
We have to show that for any $c \in \mathbb{Z}^m$, the dual of the LP $\max \{cx : x(S) \leq \text{rk}(S) \forall S \subseteq E, x \geq 0\}$ has an optimal integral solution. But this has been shown in the lecture: setting $y_e = c_i - c_{i+1}$ for any $e$ picked by the greedy algorithm, $y_e = 0$ for any other, gives an optimal dual solution that is trivially integral.

Problem 2
Let $G$ be a disconnected graph with connected components $G_1, G_2$, and let $P$ be the matching polytope of $G$, and $P_i$ the matching polytope of $G_i$, $i = 1, 2$. Show that $P = P_1 \times P_2$ (where, given polytopes $A \subseteq \mathbb{R}^n, B \subseteq \mathbb{R}^m$, we define their cartesian product $A \times B = \{(x, y) \in \mathbb{R}^{n+m} : x \in A, y \in B\}$).

Solution:
Let us denote by $E_i$ the set of edges of $G_i$, $i = 1, 2$. Notice that $M \subseteq E$ a matching in $G$ if and only if $M = M_1 \cup M_2$, where $M_i \subseteq E_i$ is a matching in $G_i$ for $i = 1, 2$. This immediately implies that $P$ and $P_1 \times P_2$ have the same vertices, hence they are the same polytope.

Problem 3
Let $C$ be the cone generated by linearly independent vectors $a_1, \ldots, a_n \in \mathbb{R}^n$, i.e.

$$C = \{\sum_{i=1}^{n} \lambda_i a_i : \lambda_i \geq 0 \forall i = 1, \ldots, n\}.$$

Show that for any $i = 1, \ldots, n$ there is a point $c \in C \cap \mathbb{Z}^n$ such that $c + e_i \in C$.

Solution:
We show that there is a point $c \in C \cap \mathbb{Z}^n$ in the cone such that a ball of radius 1 and center $c$ is entirely contained in $C$. To see this, we first show that $C$ contains arbitrary large balls. Consider the polytope $Q = \text{conv}\{\pm a_i\}$, which has dimension $n$ since the $a_i$'s are linearly independent. This means that there exist a ball of radius $r > 0$ contained in $Q$. Clearly, the translate $Q + \sum_i a_i$ is contained in $C$, and it still contains such ball. By multiplying by an arbitrary large factor $k$, we have that $k(Q + \sum_i a_i)$ is contained in $C$ and contains an arbitrary large ball $B$. Now, it is easy to see that any ball of radius at least $\sqrt{n}/2$ contains an integer point: for instance, it contains a cube with side length 1, which can be written as $v + x : 0 \leq x \leq 1$ for some translation vector $v$. Then choosing $x = [v] - v$ gives us the desired point. By requiring that $B$ has radius at least $\sqrt{n}/2 + 1$,
we are then sure that the integer point is also surrounded by a ball of radius 1 contained in \( \mathcal{B} \), hence in \( \mathcal{C} \), and we are done.

**Problem 4 \((*)\)**

Two vertices \( x^1, x^2 \) of a polyhedron \( P = \{ x \in \mathbb{R}^n : Ax \leq b \} \) are said to be *adjacent* if there is a subsystem \( A'x \leq b' \) of \( Ax \leq b \) such that \( A' \in \mathbb{R}^{(n-1) \times n} \), the rows of \( A' \) are linearly independent, and \( x^1, x^2 \) satisfy \( A'x \leq b' \) with equality.

Let \( M = (E, \mathcal{I}) \) be a matroid and \( P_M \) the corresponding matroid polytope. Given \( I_1, I_2 \in \mathcal{I} \) with \( I_1 \neq I_2 \), show that \( \chi^{I_1} \) and \( \chi^{I_2} \) are adjacent vertices of \( P_M \) if and only if one of the following conditions hold:

(i) \( I_1 \subseteq I_2 \) and \( |I_1| + 1 = |I_2| \)

(ii) \( I_2 \subseteq I_1 \) and \( |I_2| + 1 = |I_1| \)

(iii) \( |I_1 \setminus I_2| = |I_2 \setminus I_1| = 1 \) and \( I_1 \cup I_2 \notin \mathcal{I} \)

**Solution:**

Let \( A, b \) be such that \( P_M = \{ x \in \mathbb{R}^E : Ax \leq b \} \), and let \( |E| = n \).

\((\Leftarrow)\) Suppose \( I_1 \) and \( I_2 \) satisfy (i). Then \( I_1 \) and \( I_2 \) satisfy:

\[
x(e) = \text{rk}(e) \quad \forall e \in I_1
\]

\[
x(e) = 0 \quad \forall e \notin I_2,
\]

which corresponds to a submatrix of \( A \) with \( n - 1 \) linearly independent rows. The case (ii) is analogous. For the case (iii), we need a slightly different system:

\[
x(e) = \text{rk}(e) \quad \forall e \in I_1 \cap I_2
\]

\[
x(e) = 0 \quad \forall e \notin I_1 \cup I_2
\]

\[
x(I_1 \cup I_2) = \text{rk}(I_1 \cup I_2).
\]

It is immediate to see that rows of the corresponding submatrix of \( A \) are linearly independent.

\((\Rightarrow)\) First consider the case where \( I_1 \) is not a base of \( I_1 \cup I_2 \). Then by the third axiom there exists some \( j \in I_2 \setminus I_1 \) such that \( I_1 \cup j \in \mathcal{I} \). Hence

\[
\frac{1}{2} \left( \chi^{I_1} + \chi^{I_2} \right) = \frac{1}{2} \left( \chi^{I_1 \cup j} + \chi^{I_2 \setminus j} \right).
\]

We now show that this implies that \( I_1 \cup j = I_2 \) and \( I_2 \setminus j = I_1 \), meaning that we are in case (i). Indeed, since \( \chi^{I_1} \) and \( \chi^{I_2} \) are adjacent, there is a supporting hyperplane that intersects \( P_M \) exactly in the segment between \( \chi^{I_1}, \chi^{I_2} \). Consider any point in the segment (in particular the midpoint): since \( P_M \) is a polytope, it is a convex combination of vertices of \( P_M \), but it can only be expressed as combination of \( \chi^{I_1}, \chi^{I_2} \) (all other vertices are on the same side of the supporting hyperplane). Since we expressed this midpoint as the midpoint of another segment between \( \chi^{I_1 \cup j} \) and \( \chi^{I_2 \setminus j} \), it must be that these are actually the same vertices as \( \chi^{I_1} \) and \( \chi^{I_2} \), and we are done.

The case where \( I_2 \) is not a base of \( I_1 \cup I_2 \) is similar. Hence it remains to consider the case where both \( I_1 \) and \( I_2 \) are bases of \( I_1 \cup I_2 \) (hence \( I_1 \cup I_2 \notin \mathcal{I} \)). Using the strong basis exchange property, there are \( i \in I_1 \setminus I_2, j \in I_2 \setminus I_1 \) such that \( I_1 \setminus i \cup j \) and \( I_2 \setminus j \cup i \) are both bases of \( I_1 \cup I_2 \). But then

\[
\frac{1}{2} \left( \chi^{I_1} + \chi^{I_2} \right) = \frac{1}{2} \left( \chi^{I_1 \setminus i \cup j} + \chi^{I_2 \setminus j \cup i} \right)
\]

and using the same argument as before we have that \( I_1 \setminus i \cup j = I_2 \) and \( I_2 \setminus j \cup i = I_1 \), meaning that \( I_1 \) and \( I_2 \) satisfy (iii).