École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand

April 7, 2017

Discrete Optimization (Spring 2017)

Assignment 7

Problem 3 can be **submitted** until April 14 12:00 noon into the right box in front of MA C1 563.
You are allowed to submit your solutions in groups of at most three students.

Problem 1

Provide a proof or a counterexample to the following statements:

i) An iteration of the simplex method cannot move the feasible solution by a strictly positive distance while leaving the objective function value unchanged.

ii) If B is an optimal basis, then all the components of λ_B are strictly positive.

Problem 2

Let B be an optimal basis. Provide a proof or a counterexample to the following statements:

i) If λ_B is strictly positive then the optimal solution is unique.

ii) If the optimal solution is unique then λ_B is strictly positive.

Problem 3 ($\star$)

Let $\max\{c^{\mathrm{T}}x \,:\, x \in \mathbb{R}^n, Ax \leq b\}$ be a non-degenerate linear program with a unique optimal solution. Prove that λ_B is strictly positive for the corresponding optimal basis B.

Problem 4

Recall that a bounded polyhedron $P \subseteq \mathbb{R}^n$ is called a *polytope*, i.e. there exists an $M \in \mathbb{R}$ such that $P \subseteq [-M, M]^n$. In the following let P be a non-empty polytope.

(i) Prove that P has vertices.

(ii) Let $u_1, \ldots, u_\ell \in \mathbb{R}^n$ be the vertices of P. Show that P is equal to the convex hull $\mathrm{conv}(\{u_1, \ldots, u_\ell\})$, without using the separating hyperplane theorem.

Hint: To show that $P \subseteq \mathrm{conv}(\{u_1, \ldots, u_\ell\})$ one possibility is to suppose that there exists a point $x^* \in P \setminus \mathrm{conv}(\{u_1, \ldots, u_\ell\})$ and consider the following linear program and its dual:

$$\begin{aligned} \min \quad & 0^{\mathrm{T}} \cdot \lambda \\ \text{s.t.} \quad & \sum_{i=1}^{\ell} \lambda_i u_i = x^* \\ & \sum_{i=1}^{\ell} \lambda_i = 1 \\ & \lambda \geq 0 \\ & \lambda \in \mathbb{R}^\ell \end{aligned}$$

$$\begin{aligned} \max \quad & (x^*)^{\mathrm{T}} \cdot c + \beta \\ \text{s.t.} \quad & u_i^{\mathrm{T}} \cdot c + \beta \leq 0 \quad i = 1, \ldots, \ell \\ & c \in \mathbb{R}^n, \beta \in \mathbb{R} \end{aligned}$$

Conclude that the primal problem (the minimization) must be infeasible and that its dual problem (the maximization) is unbounded. From this, yield a contradiction to the fact that there always exists an optimal vertex in P for any linear optimization problem over P.

Problem 5

Given two polynomials $p_1(x) = \sum_{i=0}^{n} a_i x^i$,$p_2(x) = \sum_{i=0}^{m} b_i x^i$, we say that p_1 is lexicographically smaller than p_2, and we write $p_1 <_L p_2$, if $(a_0, a_1, \ldots, a_n) <_L (b_0, b_1, \ldots, b_m)$ (i.e. if i is the first index such that a_i is different than b_i, $a_i < b_i$). Prove that for any p_1, p_2 there is an $\epsilon^* > 0$ such that the following holds:

$$p_1(\epsilon) < p_2(\epsilon) \;\; \forall \epsilon : 0 < \epsilon < \epsilon^* \quad \Leftrightarrow \quad p_1 <_L p_2.$$