Problem 1
Provide a proof or a counterexample to the following statements:

i) An iteration of the simplex method cannot move the feasible solution by a strictly positive distance while leaving the objective function value unchanged.

ii) If \( B \) is an optimal basis, then all the components of \( \lambda_B \) are strictly positive.

Problem 2
Let \( B \) be an optimal basis. Provide a proof or a counterexample to the following statements:

i) If \( \lambda_B \) is strictly positive then the optimal solution is unique.

ii) If the optimal solution is unique then \( \lambda_B \) is strictly positive.

Problem 3 (⋆)
Let \( \max \{ c^T x : x \in \mathbb{R}^n, Ax \leq b \} \) be a non-degenerate linear program with a unique optimal solution. Prove that \( \lambda_B \) is strictly positive for the corresponding optimal basis \( B \).

Problem 4
Recall that a bounded polyhedron \( P \subseteq \mathbb{R}^n \) is called a polytope, i.e. there exists an \( M \in \mathbb{R} \) such that \( P \subseteq [-M,M]^n \). In the following let \( P \) be a non-empty polytope.

(i) Prove that \( P \) has vertices.

(ii) Let \( u_1,\ldots,u_\ell \in \mathbb{R}^n \) be the vertices of \( P \). Show that \( P \) is equal to the convex hull \( \text{conv}(\{u_1,\ldots,u_\ell\}) \), without using the separating hyperplane theorem.

Hint: To show that \( P \subseteq \text{conv}(\{u_1,\ldots,u_\ell\}) \) one possibility is to suppose that there exists a point \( x^* \in P \setminus \text{conv}(\{u_1,\ldots,u_\ell\}) \) and consider the following linear program and its dual:

\[
\begin{align*}
\min & \quad 0^T \cdot \lambda \\
\text{s.t.} & \quad \sum_{i=1}^\ell \lambda_i u_i = x^* \\
& \quad \sum_{i=1}^\ell \lambda_i = 1 \\
& \quad \lambda \geq 0 \\
& \quad \lambda \in \mathbb{R}^\ell
\end{align*}
\]

\[
\begin{align*}
\max & \quad (x^*)^T \cdot c + \beta \\
\text{s.t.} & \quad u_i^T \cdot c + \beta \leq 0 \quad i = 1,\ldots,\ell \\
& \quad c \in \mathbb{R}^n, \beta \in \mathbb{R}
\end{align*}
\]

Conclude that the primal problem (the minimization) must be infeasible and that its dual problem
(the maximization) is unbounded. From this, yield a contradiction to the fact that there always exists an optimal vertex in $P$ for any linear optimization problem over $P$.

**Problem 5**

Given two polynomials $p_1(x) = \sum_{i=0}^{n} a_i x^i, p_2(x) = \sum_{i=0}^{m} b_i x^i$, we say that $p_1$ is lexicographically smaller than $p_2$, and we write $p_1 <_L p_2$, if $(a_0, a_1, \ldots, a_n) <_L (b_0, b_1, \ldots, b_m)$ (i.e. if $i$ is the first index such that $a_i$ is different than $b_i, a_i < b_i$). Prove that for any $p_1, p_2$ there is an $\epsilon^* > 0$ such that the following holds:

$$p_1(\epsilon) < p_2(\epsilon) \quad \forall \epsilon : 0 < \epsilon < \epsilon^* \quad \Leftrightarrow \quad p_1 <_L p_2.$$