École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand
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November 25, 2016

Combinatorial Optimization (Fall 2016)

Assignment 7

Deadline: November 18 10:00, into the right box in front of MA C1 563.

Exercises marked with a $\star$ can be handed in for bonus points.

Problem 1

Given a matroid $M = (E, \mathcal{I})$ and its corresponding matroid polytope P_M, show that its associated system of inequalities $\{x(S) \leq \text{rk}(S)\ \forall S \subset E, x \geq 0\}$ is Totally Dual Integral.

Problem 2

Let G be a disconnected graph with connected components G_1, G_2, and let P be the matching polytope of G, and P_i the matching polytope of G_i, $i = 1, 2$. Show that $P = P_1 \times P_2$ (where, given polytopes $A \subset \mathbb{R}^n, B \subset \mathbb{R}^m$, we define their cartesian product $A \times B = \{(x, y) \in \mathbb{R}^{n+m} : x \in A, y \in B\}$).

Problem 3

Let $\mathcal{C}$ be the cone generated by linearly independent vectors $a_1, \ldots, a_n \in \mathbb{R}^n$, i.e.

$$\mathcal{C} = \{\sum_{i=1}^{n} \lambda_i a_i : \lambda_i \geq 0\ \forall i = 1, \ldots, n\}.$$

Show that for any $i = 1, \ldots, n$ there is a point $c \in \mathcal{C} \cap \mathbb{Z}^n$ such that $c + e_i \in \mathcal{C}$.

Problem 4 ($\star$)

Two vertices x^1, x^2 of a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ are said to be *adjacent* if there is a subsystem $A'x \leq b'$ of $Ax \leq b$ such that $A' \in \mathbb{R}^{(n-1) \times n}$, the rows of A' are linearly independent, and x^1, x^2 satisfy $A'x \leq b'$ with equality.

Let $M = (E, \mathcal{I})$ be a matroid and P_M the corresponding matroid polytope. Given $I_1, I_2 \in \mathcal{I}$ with $I_1 \neq I_2$, show that χ^{I_1} and χ^{I_2} are adjacent vertices of P_M if and only if one of the following conditions hold:

(i) $I_1 \subseteq I_2$ and $|I_1| + 1 = |I_2|$

(ii) $I_2 \subseteq I_1$ and $|I_2| + 1 = |I_1|$

(iii) $|I_1 \setminus I_2| = |I_2 \setminus I_1| = 1$ and $I_1 \cup I_2 \notin \mathcal{I}$

Hint: In the "only if" direction, you can use the following fact, known as strong basis exchange: for any two bases B_1, B_2, there are two elements $x \in B_1 \setminus B_2$ and $y \in B_2 \setminus B_1$ such that both $B_1 \setminus \{x\} \cup \{y\}$ and $B_2 \setminus \{y\} \cup \{x\}$ are bases.