École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand

March 23, 2018

Discrete Optimization (Spring 2018)

Assignment 5

Problem 5 can be **submitted** until March 30 12:00 noon into the box in front of MA C1 563. You are allowed to submit your solutions in groups of at most three students.

Problem 1

Prove that any non-empty closed, convex set is the intersection of all the half spaces which contain it.

Solution:
Let S be a non-empty closed, convex set, and let H be the intersection of all the half spaces containing S. Clearly S is contained in H. On the other hand, for any $x \notin S$, by the Separation Theorem there is a hyperplane separating x from S, i.e. there is a half space containing S and not x, hence $x \notin H$.

Problem 2

For each of the following assertions, provide a proof or a counterexample.

i) An index that has just left the basis B in the simplex algorithm cannot enter in the very next iteration.

ii) An index that has just entered the basis B in the simplex algorithm cannot leave again in the very next iteration.

Solution:

i) An index that has left the basis can enter in the very next iteration. An example is a triangle in the plane. Maybe the simplex method does not decide to walk to the neighboring optimal vertex in one step but makes a detour (while improving) via the other vertex. In this case, the inequality that has just left re-enters again.

ii) We give a first proof that uses the fact that Simplex always chooses a direction that augments the objective function. Let B be a feasible basis and let Simplex move from B to $\tilde{B} = B \setminus \{i\} \cup \{j\}$, i.e., i leaves the basis and j enters it. Note that B and $\tilde{B}$ have $n-1$ common indices. Assume that j leaves the basis in the next iteration. Let d and $\tilde{d}$ be the directions that Simplex chooses to move from B to $\tilde{B}$ and away from $\tilde{B}$, respectively. Then, $d \cdot a_k = 0 = \tilde{d} \cdot a_k$ for $k \in B \setminus \{i\}$, *i.e.*, d and $\tilde{d}$ are perpendicular to each constraint that is in the basis except a_i and a_j. Since the a_k ($k \in B \setminus \{i\}$) are $n-1$ linearly independent vectors and $d, \tilde{d} \in \mathbb{R}^n$ this means that $d \parallel \tilde{d}$. Since j entered the basis, this means that $a_j^{\mathrm{T}} d > 0$. Since j leaves the basis in the next step $a_j^{\mathrm{T}} \tilde{d} = -1$. Thus, $d = -\omega\ \tilde{d}$ for some constant $\omega > 0$. In particular, this means that the Simplex is moving in the opposite direction. Now, due to the choice of the direction in Simplex we know that $c^{\mathrm{T}} d > 0$ and $c^{\mathrm{T}} \tilde{d} > 0$. But, this yields a contradiction since $0 < c^{\mathrm{T}} \tilde{d} = -\omega\ c^{\mathrm{T}} d < 0$.

We give a second proof showing that λ_j is non-negative in the iteration succeeding the entrance of j into the basis, i.e., j cannot leave the basis in the next iteration. In the Simplex iteration at $\tilde{B}$ we have $\lambda^{\mathrm{T}} A_{\tilde{B}} = c^{\mathrm{T}}$. For ease of notation, assume that $\tilde{B} = \{a_1, \ldots, a_{n-1}, \tilde{a}\}$ where $\tilde{a} = a_j$. If $\lambda_n = 0$, then $\tilde{a}$ can not leave the basis in the next iteration (for this, λ_n would have to be strictly negative). Hence, we can assume $\lambda_n \neq 0$. Then, rewriting the above yields

$$\lambda_1 a_1 + \cdots + \lambda_{n-1} a_{n-1} + \lambda_n \tilde{a} = c \quad \Leftrightarrow \quad \frac{-\lambda_1}{\lambda_n} a_1 + \cdots + \frac{-\lambda_{n-1}}{\lambda_n} a_{n-1} + \frac{1}{\lambda_n} c = \tilde{a}.$$

Since j entered the basis we know $\tilde{a}^{\mathrm{T}} d > 0$ (d is still the direction from B to $\tilde{B}$). This gives

$$\begin{aligned} 0 < \tilde{a}^{\mathrm{T}} d &= \frac{-\lambda_1}{\lambda_n} a_1^{\mathrm{T}} d + \cdots + \frac{-\lambda_{n-1}}{\lambda_n} a_{n-1}^{\mathrm{T}} d + \frac{1}{\lambda_n} c^{\mathrm{T}} d \\ &= 0 + \cdots + 0 + \frac{1}{\lambda_n} c^{\mathrm{T}} d. \end{aligned}$$

Since $c^{\mathrm{T}} d > 0$ this implies $\lambda_n > 0$ which means that j can not leave the basis in the very next iteration.

Problem 3

Given the following linear program:

$$\begin{aligned} \max \quad & a + 3b \\ \text{s.t.} \quad & a + b \leq 2 && (1) \\ & a \leq 1 && (2) \\ & -a \leq 0 && (3) \\ & -b \leq 0 && (4) \end{aligned}$$

Solve it with the Simplex method starting with the initial feasible basic solution induced by the constraints (2) and (4). For each iteration indicate the current basis and the corresponding vertex, λ_B, the direction in which the Simplex moves and how far it moves. At the end indicate the optimal objective value and the proof of optimality (i.e. the final λ).

Solution:

We give the result of the Simplex calculations below:

iteration	basis	vertex	λ	direction	ϵ	index exchange
1	$\{2,4\}$	$(1,0)^T$	$(1,-3)^T$	$(0,1)^T$	1	$4 \Rightarrow 1$
2	$\{1,2\}$	$(1,1)^T$	$(3,-2)^T$	$(-1,1)^T$	1	$2 \Rightarrow 3$
3	$\{1,3\}$	$(0,2)^T$	$(3,2)^T$			

The last vertex is the optimal solution with an objective function value of 6. It is best to also check this with a drawing.

Problem 4

Provide a proof or counterexample to the following statements:

i) An iteration of the simplex method cannot move the feasible solution by a strictly positive distance while leaving the objective function value unchanged.

ii) If B is an optimal basis, then all the components of λ_B are strictly positive.

Solution:

We consider the standard linear program $\max\{c^{\mathrm{T}} x \ : \ x \in \mathbb{R}^n, Ax \leq b\}$.

i) True. Let x be the current feasible basic solution and $x' = x + \epsilon d$ be the next feasible basic solution when the Simplex method moves from x in the direction d. Since it moves a strictly positive distance we have $\epsilon > 0$. The Simplex chooses d such that $c^{\mathrm{T}}d > 0$. This implies that $c^{\mathrm{T}}x' = c^{\mathrm{T}}x + \epsilon c^{\mathrm{T}}d > c^{\mathrm{T}}x$.

ii) False. Consider the linear program

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 1 \\ & -x_1 \leq 0 \\ & -x_2 \leq 0 \end{aligned}$$

The feasible region of this LP is the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(1,0)$, and $(0,1)$. An optimal basis is $\{1,2\}$ with corresponding feasible basic solution $(0,1)$. Here,

$$\lambda^{\mathrm{T}} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = [1 \quad 1]$$

which implies $\lambda^{\mathrm{T}} = [1 \quad 0]$.

Problem 5 ($\star$)

A polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ *contains a line*, if there exists a nonzero $v \in \mathbb{R}^n$ and an $x^* \in \mathbb{R}^n$ such that for all $\lambda \in \mathbb{R}$, the point $x^* + \lambda \cdot v \in P$. Show that a nonempty polyhedron P contains a line if and only if A does not have full column-rank.

Solution:

Assume that P contains a line $\{x \in \mathbb{R}^n : x = x^* + \lambda \cdot v,\ \lambda \in \mathbb{R}\}$. We claim that $v \in ker(A)$, i.e. for all rows a_i of A we have $a_i^T v = 0$. Assume for contradiction that there is a row a_i with $a_i^T v \neq 0$. Then we can choose $\lambda \in \mathbb{R}$ such that $a_i^T x^* + \lambda a_i^T v > b_i$ (namely such that $|\lambda| > |\frac{b_i - a_i^T x^*}{a_i^T v}|$). Thus for $x := x^* + \lambda v$ we have $x \notin P$ because $a_i^T x > b_i$. This is a contradiction to the fact that P contains the line $\{x \in \mathbb{R}^n : x = x^* + \lambda \cdot v,\ \lambda \in \mathbb{R}\}$.
Thus the kernel of A is not empty, and A does not have full column rank.
Conversely, if A does not have full column rank, let x^* be some feasible point of the polyhedron, and let v be a nonzero vector from the kernel of A. Then $x^* + \lambda \cdot v \in P$ for all $\lambda \in \mathbb{R}$. Hence P contains a line.

Problem 6

Consider the following linear program

$$\max\{c^T x : Ax \leq b,\ x \geq 0\}. \tag{5}$$

Suppose that we re-write the constraints $Ax \leq b$ as $A_1 x \leq b_1$ and $A_2 x \leq b_2$ with $b_1 \geq 0$ and $b_2 < 0$, and consider the linear program:

$$\min\{\mathbf{1}^T y : A_1 x \leq b_1,\ A_2 x \leq b_2 + y,\ x, y \geq 0, y \leq -b_2\}. \tag{6}$$

1. Prove that (5) is feasible if and only if (6) has optimal value 0.

2. Assume that (6) has optimal value 0, and let B be an optimal basis for (6). Then show that B without the indices corresponding to $y \geq 0$ is a feasible basis for the linear program (5).

Solution:
Let n be the dimension of x and m be the dimension of y, which is also the length of b_2.

1. For $y \geq 0$, $\mathbf{1}^T y = 0$ if and only if $y = 0$. Hence (6) has optimal value equal to 0 if and only if there is $(x, 0) \in \mathbb{R}^m$ such that $A_1 x \leq b_1, A_2 x \leq b_2 + 0$ and $x \geq 0$, i.e. if and only if (5) is feasible.

2. The submatrix A_B corresponding to B has form

$$\begin{pmatrix} A_1' & \mathbf{0} \\ A_2' & -I' \\ -I'' & \mathbf{0} \\ \mathbf{0} & -I_m \end{pmatrix},$$

where A_1', A_2' are submatrices of A_1, A_2 respectively, $\mathbf{0}$ indicates a matrix of zeros, I_m is the $m \times m$ identity matrix, I', I'' are submatrices of I_m, I_n respectively. A_B is non-singular and has size $(n+m) \times (n+m)$. Let B' be the basis obtained from B removing the indices corresponding to $y \geq 0$, and consider the submatrix of coefficients of (5) corresponding to B': $A_{B'} = \begin{pmatrix} A_1' \\ A_2' \\ -I'' \end{pmatrix}$. $A_{B'}$ has size $n \times n$ and is non-singular (to see this, compute the determinant of A_B using Laplace expansion), hence B' is a basis of (5). Now, let $(x, 0) = A_B^{-1} b_B$, thanks to part 1. x is a feasible point of (5) satisfying $A_1' x = b_1, A_2' x = b_2$, hence B' is a feasible basis for (5).