École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand
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October 28, 2016

Combinatorial Optimization (Fall 2016)

Assignment 5

Deadline: November 11 10:00, into the right box in front of MA C1 563.

Exercises marked with a $\star$ can be handed in for bonus points.

Problem 1

In class we saw that we can decide whether a graph G has a perfect matching or not looking at the determinant of the Tutte matrix A_G, which is a polynomial with variables $x_e, e \in E(G)$. In this exercise we will see a randomized approach to check whether a polynomial is identically 0 or not. This algorithm would then allow us to decide whether G has a perfect matching or not.

1. Prove the following (Schwartz-Zippel Lemma): let p be a polynomial in variables $x_1, \ldots, x_n$ of total degree d. Assume p is not identically 0, and let $S \subset \mathbb{R}$ be any finite set. If $y_1, \ldots, y_n$ are chosen independently and uniformly at random from S, then:

$$\Pr[p(y_1, \ldots, y_n) = 0] \leq \frac{d}{|S|}$$

   Hint: use induction on n.

2. Use part 1. to derive a randomized algorithm that takes a polynomial p as an input and returns $p \equiv 0$ or $p \not\equiv 0$. The algorithm should have one-sided error: if it returns $p \not\equiv 0$, then it is correct; if it returns $p \equiv 0$, then the probability that $p \not\equiv 0$ can be made arbitrarily small.

Problem 2 ($\star$)

1. Prove that if M_1 and M_2 are matchings of G and $|M_2| > |M_1|$ then there exists at least $|M_2| - |M_1|$ vertex-disjoint M_1-augmenting paths.

2. Prove that if M is a matching of G that is not maximum cardinality then there exists a maximum cardinality matching M^* such that every vertex covered by M is also covered by M^*. *Hint: use part 1.*

Problem 3 ($\star$)

Suppose you are given an oracle that given a graph G, tells you whether G has a perfect matching or not. Show how to use this oracle to determine the maximum cardinality matching of G.
Hint: you should modify the graph at each call of the oracle. The total number of calls should be at most $|V| + |E|$.