École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand

March 23, 2018

Discrete Optimization (Spring 2018)

Assignment 5

Problem 5 can be **submitted** until March 30 12:00 noon into the box in front of MA C1 563. You are allowed to submit your solutions in groups of at most three students.

Problem 1

Prove that any non-empty closed, convex set is the intersection of all the half spaces which contain it.

Problem 2

For each of the following assertions, provide a proof or a counterexample.

i) An index that has just left the basis B in the simplex algorithm cannot enter in the very next iteration.

ii) An index that has just entered the basis B in the simplex algorithm cannot leave again in the very next iteration.

Problem 3

Given the following linear program:

$$\begin{aligned} \max \quad & a + 3b & \\ \text{s.t.} \quad & a + b \leq 2 & (1) \\ & a \leq 1 & (2) \\ & -a \leq 0 & (3) \\ & -b \leq 0 & (4) \end{aligned}$$

Solve it with the Simplex method starting with the initial feasible basic solution induced by the constraints (2) and (4). For each iteration indicate the current basis and the corresponding vertex, λ_B, the direction in which the Simplex moves and how far it moves. At the end indicate the optimal objective value and the proof of optimality (i.e. the final λ).

Problem 4

Provide a proof or counterexample to the following statements:

i) An iteration of the simplex method cannot move the feasible solution by a strictly positive distance while leaving the objective function value unchanged.

ii) If B is an optimal basis, then all the components of λ_B are strictly positive.

Problem 5 ($\star$)

A polyhedron $P = \{x \in \mathbb{R}^n \colon Ax \leq b\}$ *contains a line*, if there exists a nonzero $v \in \mathbb{R}^n$ and an $x^* \in \mathbb{R}^n$ such that for all $\lambda \in \mathbb{R}$, the point $x^* + \lambda \cdot v \in P$. Show that a nonempty polyhedron P contains a line if and only if A does not have full column-rank.

Problem 6

Consider the following linear program

$$\max\{c^T x \colon Ax \leq b,\, x \geq 0\}. \tag{5}$$

Suppose that we re-write the constraints $Ax \leq b$ as $A_1 x \leq b_1$ and $A_2 x \leq b_2$ with $b_1 \geq 0$ and $b_2 < 0$, and consider the linear program:

$$\min\{\mathbf{1}^T y \colon A_1 x \leq b_1,\, A_2 x \leq b_2 + y,\, x, y \geq 0, y \leq -b_2\}. \tag{6}$$

1. Prove that (5) is feasible if and only if (6) has optimal value 0.
2. Assume that (6) has optimal value 0, and let B be an optimal basis for (6). Then show that B without the indices corresponding to $y \geq 0$ is a feasible basis for the linear program (5).