Prof. F. Eisenbrand EPFL - DISOPT

Combinatorial Optimization

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Sheet 5 November 17, 2011

General remark:
In order to obtain a bonus for the final grading, you may hand in written solutions to the exercises marked with a star at the beginning of the exercise session on November 29.

Exercise 1

Let $M = (E, \mathcal{I})$ be an independence system (i.e. $(M0)$ and $(M1)$ hold).
Prove or give a counterexample that the following two properties are equivalent:

$(M2)$ For all $I, J \in \mathcal{I}$ and $|J| > |I|$ there exists some $e \in J \backslash I$ such that $I \cup \{e\} \in \mathcal{I}$.

$(M2')$ For all $I, J \in \mathcal{I}$ with $I \cap J = \emptyset$ and $|J| > |I|$, there exists some $e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{I}$.

Exercise 2

Let $G = (V, E)$ be a graph. Show that $\operatorname{conv}(\{\chi_F \in \{0,1\}^{|E|} \colon F \text{ is a forest in } G\})$, the convex hull of incidence vectors of all forests in G is equal to the following set

$$\{x \in \mathbb{R}^{|E|}_{\geq 0} \colon \sum_{e \in E[T]} x(e) \leq |T| - 1, \text{ for all } \emptyset \neq T \subseteq V\}$$

Exercise 3 ($\star$)

Let E be a finite set and let $r : 2^E \to \mathbb{Z}_+$. Then r is the rank function of a matroid $(E, \mathcal{I})$ if and only if for all $I, J \subseteq E$:

(i) $r(I) \leq r(J) \leq |J|$ if $I \subseteq J$,

(ii) $r(I \cap J) + r(I \cup J) \leq r(I) + r(J)$.

Hint: "$\Leftarrow$" Show first that $(E, \mathcal{I})$ where $\mathcal{I} = \{I \subseteq E \colon |I| = r(I)\}$ is a matroid (using condition (iii) in exercise 2 on sheet 4). Conclude that r is indeed a rank function by induction on the size of $A \subseteq E$.

Exercise 4

We consider the following generalization of the matroid polytope:
Let E be a finite set and let $f : 2^E \to \mathbb{R}_+$ be a submodular function, i.e. $f(I \cap J) + f(I \cup J) \leq f(I) + f(J)$ for all $I, J \subseteq E$. The **polymatroid** defined by f is the polytope

$$P(f) := \left\{ x \in \mathbb{R}^E \colon x \geq 0, \sum_{e \in A} x_e \leq f(A) \qquad \forall A \subseteq E \right\}$$

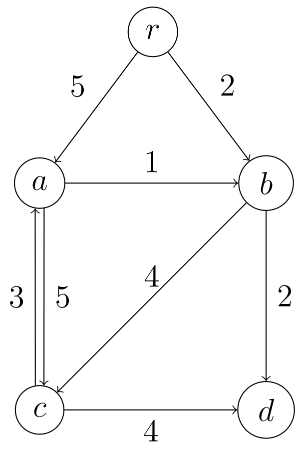
Note that the matroid polytope is a special case where f is the rank function of a matroid over E. What is the separation problem for a polymatroid?

Let $G = (V, E)$ be a graph and $c : E \to \mathbb{R}_+$. Consider the function $f(A) := c(\delta(A)) = \sum_{e \in \delta(A)} c(e)$ for all $A \subseteq V$.

(i) Show that f is submodular.
(ii) Show how to solve the separation problem for the polymatroid defined by f over V in time $O(m^2 n^2)$.
(iii) Conclude how to maximize a weight function $w : V \to \mathbb{R}$ over the polymatroid $P(f)$.

Exercise 5

Trace the steps of algorithm from the lecture to compute a minimum weight arborescence rooted at r in the following example.



Exercise 6 ($\star$)

Let $D = (V, A)$ be a directed graph with root $r \in V$. Suppose that D does not contain an arborescence rooted at r. Prove that there exists a (nonempty) strongly connected component K in D such that $r \notin K$ and $|\delta^{in}(K)| = 0$.