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Due Date: May 06, 2010
Discussions: March 15, April 22, April 29

Discrete Optimization

Spring 2010

Assignment Sheet 4

You can hand in written solutions for up to two of the exercises marked with (*) or (Δ) to obtain bonus points. The duedate for this is May 06, 2010, before the exercise session starts. Math students are restricted to exercises marked with (*). Non-math students can choose between (*) and (Δ) exercises.

Exercise 1 (Δ)
Let $M \in \mathbb{Z}^{n \times m}$ be totally unimodular. Prove that the following matrices are totally unimodular as well:

1. M^T
2. $(M \quad I_n)$
3. $(M \quad -M)$
4. $M \cdot (I_n - 2e_j e_j^T)$ for some j

I_n is the $n \times n$ identity matrix, and e_j is the vector having a 1 in the jth component, and 0 in the other components.

Exercise 2 (*)
A family $\mathscr{F}$ of subsets of a finite groundset E is *laminar*, if for all $C, D \in \mathscr{F}$, one of the following holds:

$$(i)\ C \cap D = \emptyset,\ \ (ii)\ C \subseteq D,\ \ (iii)\ D \subseteq C.$$

Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two laminar families of the same groundset E and consider its union $\mathscr{F}_1 \cup \mathscr{F}_2$. Define the $|\mathscr{F}_1 \cup \mathscr{F}_2| \times |E|$ adjacency matrix A as follows: For $F \in \mathscr{F}_1 \cup \mathscr{F}_2$ and $e \in E$ we have $A_{F,e} = 1$, if $e \in F$ and $A_{F,e} = 0$ otherwise.

Show that A is totally unimodular.

Exercise 3
Consider the following scheduling problem: Given n tasks with periods $p_1, \ldots, p_n \in \mathbb{N}$, we want to find offsets $x_i \in \mathbb{N}_0$, such that every task i can be executed periodically at times $x_i + p_i \cdot k$ for all $k \in \mathbb{N}_0$. In other words, for all pairs i, j of tasks we require $x_i + k \cdot p_i \neq x_j + l \cdot p_j$ for all $k, l \in \mathbb{N}_0$.

Formulate the problem of finding these offsets as an integer program (with zero objective function).

Exercise 4 (*)
Let $P = \{x \in \mathbb{R}^n : \; Ax \le b\}$ be a polyhedron. Show that the following are equivalent for a feasible x^*:

i) x^* is a vertex of P.

ii) There exists a set $B \subseteq \{1,\dots,m\}$ such that $|B| = n$, A_B is invertible and $A_B x^* = b_B$. Here the matrix A_B and the vector b_B consists of the rows of A indexed by B and the components of b indexed by B respectively.

iii) For every feasible $x_1, x_2 \in P$, $x_1 \ne x^* \ne x_2$, one has $x^* \notin \operatorname{conv}\{x_1, x_2\}$.

Exercise 5
Show the following: A polyhedron $P \subseteq \mathbb{R}^n$ with vertices is integral, if and only if each vertex is integral.

Exercise 6
Consider the polyhedron $P = \{x \in \mathbb{R}^3 : \; x_1 + 2\,x_2 + 4\,x_3 \le 4, \; x \ge 0\}$. Show that this polyhedron is integral.

Exercise 7
Which of these matrices is totally unimodular? Justify your answer.

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Exercise 8
Consider the complete graph G_n with 3 vertices, i.e., $G = (\{1,2,3\}, \binom{3}{2})$. Is the polyhedron of the linear programming relaxation of the vertex-cover integer program integral?