École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand
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October 21, 2016

Combinatorial Optimization (Fall 2016)

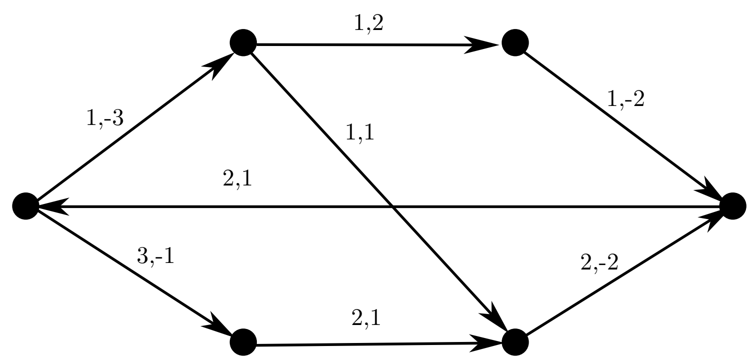
Assignment 4

Deadline: October 28 10:00, into the right box in front of MA C1 563.

Exercises marked with a $\star$ can be handed in for bonus points.

Problem 1

Determine in the following networks a circulation of minimum cost using the negative cycle canceling algorithm (the numbers on the arcs stand for (capacity, cost)):

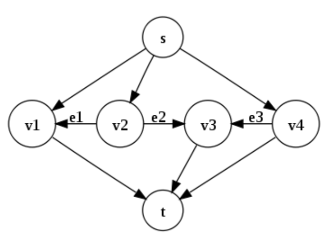


Solution:
The final circulation has cost -4.

Problem 2 ($\star$)

1. Prove that, in a directed graph with integral capacities, the negative cycle canceling algorithm always terminates.

2. Consider the following graph:



where the capacities of e_1, e_2, e_3 are 1, 1, $r = \frac{\sqrt{5}-1}{2}$, (note that r satisfies $r^2 = 1 - r$), and the other edges have as capacity some integer $M \geq 2$. Show that the Ford Fulkerson algorithm does not terminate on this graph if the wrong paths are chosen. You should start with the path sv_2v_3t and then choose the following paths in the right order: $P_A = sv_1v_2v_3t$, $P_B = sv_4v_3v_2v_1t$, $P_C = sv_2v_3v_4t$, so that flow can be pushed through the sequence of paths infinitely many times. Does the flow given by this procedure converge to the maximum flow (which is $2M + 1$)?

3. Give an example of a graph with real capacities such that negative cycle canceling algorithm does not terminate. (Hint: transform the example from above in a minimum cost circulation instance, adding extra arcs and giving costs).

Solution:

1. In the negative cycle canceling algorithm, we start with a circulation that is 0 everywhere and at every iteration we augment it of the value of the bottleneck capacity, which is integral and strictly positive. Hence the circulation remains integral throughout the algorithm, and its cost strictly decreases every iteration (since we augment along negative cycles). Now, there is only a finite number of different integral flows under the capacities (since they are finite) and we cannot use the same flow twice as the cost decreases at each iteration, hence the number of iteration must be finite, i.e. the algorithm terminates after a finite number of steps.

2. The correct sequence to choose is: P_B, P_C, P_B, P_A. The table below shows how the capacities of the horizontal edges and the flow varies during the first application of the sequence:

$c(e_1)$	$c(e_2)$	$c(e_3)$	$v(f)$
1	1	r	0
1	0	r	1
r^2	r	0	$1+r$
r^2	0	r	$1+2r$
0	r^2	r^3	$1+2r+r^2$
r^2	0	r^3	$1+2r+2r^2$

   We now generalize to the k-th application of the sequence. Suppose the horizontal residual capacities are r^{k-1}, 0, and r^k for some $k \geq 1$.

   (a) Augment along P_B, adding r^k to the flow; the residual capacities are now r^{k+1}, r^k, 0.

   (b) Augment along P_C, adding r^k to the flow; the residual capacities are now r^{k+1}, 0, r^k.

   (c) Augment along P_B, adding r^{k+1} to the flow; the residual capacities are now 0, r^{k+1}, r^{k+2}.

   (d) Augment along P_A, adding r^{k+1} to the flow; the residual capacities are now r^{k+1}, 0, r^{k+2}.

   Thus, after $4n+1$ augmentation steps, the residual capacities are r^{2n-2}, 0, r^{2n-1}. As the number of augmentation steps grows to infinity, the value of the flow converges to

$$1 + 2\sum_{k=1}^{\infty} r^k = 1 + \frac{2}{1-r} - 2 = 2 + \sqrt{5},$$

   even though the maximum flow value is clearly $2M+1$.

3. Simply add an arc ts, of capacity M and cost -1. The other costs are 0. Now, each $s-t$ path in the original network yields a negative cycle, and choosing the same sequence of paths as above gives a new negative cycle to cancel at every iteration, hence the negative cycle canceling algorithm does not terminate.

Problem 3

Given a directed graph D with capacities c and costs w on the arcs, a feasible $s-t$ flow f of D is *extreme* if for any $s-t$ flow f' with the same value as f, $w(f') \geq w(f)$. For some value $M \in \mathbb{R}_+$, we want to know if there exist an extreme $s-t$ flow of value M. Model this problem as a minimum cost circulation problem.

Solution:
Simply add an arc ts (we know we can assume for simplicity that no arc ts was present in D), with capacity M and cost negative enough (say, $-\sum_a |w(a)| - 1$). Then there is a correspondence between $s-t$ flows on D of value M and circulations on $D + ts$ with value M on the arc ts (the cost of the flow is clearly equal to the cost of the circulation minus $w(ts)$). Moreover, a circulation filling up the capacity on ts has smaller cost than any other. Hence, if there is an $s-t$ flow of value M, the minimum cost circulation will yield such a flow of minimum cost, hence an extreme flow.