Problem 1
Consider the polyhedron:

\[ P = \begin{cases} 
 x_1 + 2x_2 + x_3 \leq 5 \\
 3x_1 + x_2 + x_3 \leq 3 \\
 x_1 \leq 1 \\
 x_1 + x_2 \leq 2 \\
 x_2 + x_3 \leq 3 \\
 x_1 \geq 0 \\
 x_1 + x_2 \geq 0 \\
 x_2 + x_3 \geq 0 
\end{cases} \]

State which of the following points are vertices of \( P \): \( p_0 = (0, 0, 3) \), \( p_1 = (0, 1, 1) \), \( p_2 = (1, 4, -4) \), \( p_3 = (1/2, 3/2, 0) \), \( p_4 = (1, -1, 1) \).

Solution:
For each point \( p \), we need to check whether the submatrix of the inequalities that \( p \) satisfies with equality has full rank (i.e. equal to 3), and whether \( p \) is in \( P \). Proceeding this way, we see that only \( p_0 \) and \( p_4 \) are vertices.

Problem 2
Let \( A \in \mathbb{R}^{n \times n} \) be a non-singular matrix and let \( a_1, \ldots, a_n \in \mathbb{R}^{n} \) be the columns of \( A \).

i) Show that \( \text{cone}(\{a_1, \ldots, a_n\}) \) is the polyhedron \( P = \{x \in \mathbb{R}^n : A^{-1}x \geq 0\} \).

ii) Show that \( \text{cone}(\{a_1, \ldots, a_k\}) \) for \( k \leq n \) is the set

\[ P_k = \{x \in \mathbb{R}^n : a_i^{-1}x \geq 0, i = 1, \ldots, k, a_k^{-1}x = 0, i = k+1, \ldots, n\}, \]

where \( a_i^{-1} \) denotes the \( i \)-th row of \( A^{-1} \).

Solution:

i) We obtain the following (where \([n]\) denotes the set \(\{1, 2, \ldots, n\}\)):

\[ \text{cone}(\{a_1, \ldots, a_n\}) = \{x = \sum_{i \in [n]} \lambda_i a_i : \lambda_i \in \mathbb{R}_{\geq 0} \forall i \in [n]\} = \{x = A\lambda : \lambda \in \mathbb{R}^{n}_{\geq 0}\} = \{x \in \mathbb{R}^n : A^{-1}x = \lambda, \lambda \geq 0\} = \{x \in \mathbb{R}^n : A^{-1}x \geq 0\}. \]

ii) Analogously one has:

\[ \text{cone}(\{a_1, \ldots, a_k\}) = \{x = A\lambda : \lambda \in \mathbb{R}^{n}_{\geq 0}, \lambda_i = 0 \text{ for } i > k\} = \{x \in \mathbb{R}^n : A^{-1}x = \lambda, \lambda \geq 0, \lambda_i = 0 \text{ for } i > k\} = \{x \in \mathbb{R}^n : a_i^{-1}x \geq 0, i = 1, \ldots, k, a_k^{-1}x = 0, i = k+1, \ldots, n\}. \]
Problem 3  
Prove the following variant of Farkas' lemma: Let $A \in \mathbb{R}^{m \times n}$ be a matrix and $b \in \mathbb{R}^m$ be a vector. The system $Ax \leq b$, $x \in \mathbb{R}^n$ has a solution if and only if for all $\lambda \in \mathbb{R}_{\geq 0}^m$ with $\lambda^T A = 0$ one has $\lambda^T b \geq 0$. Hint: Use the version of Farkas’ lemma in the lecture notes, Theorem 3.11

Solution:
The system $Ax \leq b, x \in \mathbb{R}^n$ is feasible if and only if the system $A(x^+ - x^-) + s = b$ has a solution $\bar{x} = [x^+ \ x^- \ s]^T \geq 0$, where $x^+, x^- \in \mathbb{R}^n$ and $s \in \mathbb{R}^m$. We could rewrite the latter system as $\bar{A}\bar{x} = b$ with $\bar{A} = [A - A I_m]$. By applying the Farkas’ lemma seen in class to this new system we obtain that the original system $Ax \leq b$ is feasible if and only if for all $\lambda \in \mathbb{R}^m$ such that $\lambda^T [A - A I_m] \geq 0 \iff \lambda^T A \geq 0, \lambda^T (-A) \geq 0 \iff \lambda^T A = 0$ and $\lambda^T I_m = \lambda^T \geq 0$ one has $\lambda^T b \geq 0$.

Problem 4

Consider the vectors

$x_1 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, x_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, x_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, x_5 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$.

The vector

$v = x_1 + 3x_2 + 2x_3 + x_4 + 3x_5 = \begin{pmatrix} 15 \\ 5 \\ 31 \end{pmatrix}$

is a conic combination of the $x_i$.

Write $v$ as a conic combination using only three vectors of the $x_i$.

Hint: Recall the proof of Carathéodory’s theorem

Solution:
We notice that: $4x_1 - 5x_3 - x_4 = 0$, hence we can write

$v = x_1 + 3x_2 + 2x_3 + x_4 + 3x_5 - \epsilon(4x_1 - 5x_3 - x_4) = (1 - 4\epsilon)x_1 + 3x_2 + (2 + 5\epsilon)x_3 + (1 - \epsilon)x_4 + 3x_5$.

We choose $\epsilon = 1/4$ to obtain:

$v = 3x_2 + \frac{13}{4}x_3 + \frac{5}{4}x_4 + 3x_5$

We now repeat the procedure, using $x_2 + x_3 - x_4 - x_5 = 0$, and finally we get:

$v = \frac{1}{4}x_2 + \frac{17}{4}x_4 + 6x_5$

Problem 5

Consider the following classification problem: we are given $p_1, \ldots, p_N$ points in $\mathbb{R}^d$, and each point is colored either blue or red. We want to determine if there is an hyperplane $\alpha = \{ax = b\}$ that strictly separates the blue points from the red ones (i.e. such that $ap_i > b$ for all blue points and $ap_i \leq b$ for all red points) and, in case of a positive answer, find such $\alpha$. Show how to solve this problem using linear programming.

Solution:
Consider the following linear program (notice that $a \in \mathbb{R}^d$, $b \in \mathbb{R}$ are variables):

$$\max \epsilon \quad \text{s.t.} \quad ap_i \geq 1 + \epsilon \quad \forall \ p_i \text{ blue}$$

$$ap_i \leq 1 \quad \forall \ p_i \text{ red}$$

$$\epsilon \geq 0$$
If a separating hyperplane exists, then (by changing the sign of the coefficients or by slightly translating it) we can write it as \( ax = 1 \) and if \( \epsilon = \min \{ ap_i : p_i \text{ is blue} \} \), we have that \((a, \epsilon)\) is a feasible solution to the linear program with positive objective value. On the other hand, if there is a feasible solution with positive objective value, then the corresponding hyperplane \( ax = 1 \) strictly separates the blue points from the red.

**Problem 6 (\( \star \))**

Prove that for a finite set \( X \subseteq \mathbb{R}^n \) the conic hull \( \text{cone}(X) \) is closed and convex.

**Hint:** Use Problem 2 and Carathéodory’s theorem: Let \( X \subseteq \mathbb{R}^n \), then for each \( x \in \text{cone}(X) \) there exists a set \( \tilde{X} \subseteq X \) of cardinality at most \( n \) such that \( x \in \text{cone}(\tilde{X}) \). The vectors in \( \tilde{X} \) are linearly independent.

**Solution:**

Denote with \( A_X \) the matrix whose columns are vectors in \( X \), and analogously with \( A_{\tilde{X}} \) the one corresponding to a set \( \tilde{X} \). Points \( u, v \in \text{cone}(X) \) can then be written as \( u = A_X \lambda_u \) and \( v = A_X \lambda_v \) for some vectors \( \lambda_u, \lambda_v \geq 0 \). Furthermore point \( p = \gamma u + (1 - \gamma)v \) for some \( \gamma \in [0, 1] \) can be written as \( p = \gamma A_X \lambda_u + (1 - \gamma)A_X \lambda_v = A_X (\gamma \lambda_u + (1 - \gamma)\lambda_v) \) so \( p \in \text{cone}(X) \) since the vector \( \gamma \lambda_u + (1 - \gamma)\lambda_v \geq 0 \). This proves that \( \text{cone}(X) \) is a convex set.

In order to see that \( \text{cone}(X) \) is closed we need to show that for every convergent sequence \( (y_k)_{k \in \mathbb{N}} \) where \( y_k \in \text{cone}(X) \) we have that \( y = \lim_{k \to \infty} y_k \) belongs to \( \text{cone}(X) \). By Carathéodory’s theorem we know that for every \( y_k \) there is a set \( \tilde{X}_k \subseteq X \) of at most \( n \) linearly independent vectors such that \( y_k \in \text{cone}(\tilde{X}_k) \). Since there are only finitely many such subsets \( \tilde{X} \), there is one of them such that \( \tilde{X} = \tilde{X}_k \) for infinitely many \( k \), hence we can restrict our sequence only to those values of \( k \), which we denote by \( k_1, k_2, \ldots \). The restricted subsequence \( (y_{k_i})_{i \in \mathbb{N}} \) satisfies \( y_{k_i} \in \text{cone}(\tilde{X}) \) for every \( i \) and has the same limit \( y \). We now claim that \( y \in \text{cone}(\tilde{X}) \), which concludes the proof as \( \text{cone}(\tilde{X}) \subseteq \text{cone}(X) \). Notice that this is equivalent to showing that \( \text{cone}(\tilde{X}) \) is closed. Let \( k = |\tilde{X}| \leq n \), and let \( A \) be a non-singular matrix formed by \( A_{\tilde{X}} \) (whose columns are linearly independent) and \( n - k \) other columns. Applying Problem 1.ii to \( A \), it follows that \( \text{cone}(\tilde{X}) = P_k \), which is closed as it is intersection of half-spaces (which are closed sets).

**Alternative proof that \( \text{cone}(X) \) is closed:** We claim that

\[
\text{cone}(X) = \bigcup_{\tilde{X} \subseteq X \text{ lin.ind.}} \text{cone}(\tilde{X}).
\]

By the previous exercise we have that if the vectors in \( \tilde{X} \) are linearly independent, \( \text{cone}(\tilde{X}) \) is a polyhedron and thus it is closed (any polyhedron is the intersection of finitely many half spaces, which are closed sets). Since \( X \) is a finite set the number of subsets of \( X \) is also finite and thus \( \bigcup_{\tilde{X} \subseteq X \text{ lin.ind.}} \text{cone}(\tilde{X}) \) is a finite union of closed sets, hence it is closed.

We now prove the claim. The "\( \supseteq \)" direction trivially follows from \( \tilde{X} \subseteq X \) and the conic hull definition. In order to prove "\( \supseteq \)" let \( x \in \text{cone}(X) \). Then, by Caratheodory’s theorem there exists a linearly independent set \( \tilde{X} \subseteq X \) such that \( x \in \text{cone}(\tilde{X}) \), which concludes the proof.