École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand

March 16, 2018

Discrete Optimization (Spring 2018)

Assignment 4

Problem 6 can be **submitted** until March 23 12:00 noon into the box in front of MA C1 563. You are allowed to submit your solutions in groups of at most three students.

Problem 1

Consider the polyhedron:

$$P = \begin{cases} x_1 + 2x_2 + x_3 & \leq & 5 \\ 3x_1 + x_2 + x_3 & \leq & 3 \\ x_1 & \leq & 1 \\ x_1 + x_2 & \leq & 2 \\ x_2 + x_3 & \leq & 3 \\ x_1 & \geq & 0 \\ x_1 + x_2 & \geq & 0 \\ x_2 + x_3 & \geq & 0 \end{cases}$$

State which of the following points are vertices of P: $p_0 = (0, 0, 3)$, $p_1 = (0, 1, 1), p_2 = (1, 4, -4)$, $p_3 = (1/2, 3/2, 0), p_4 = (1, -1, 1)$.

Problem 2

Let $A \in \mathbb{R}^{n\times n}$ be a non-singular matrix and let $a_1, \dots, a_n \in \mathbb{R}^n$ be the columns of A.

i) Show that $\text{cone}(\{a_1, \dots, a_n\})$ is the polyhedron $P = \{x \in \mathbb{R}^n : A^{-1}x \geq 0\}$.

ii) Show that $\text{cone}(\{a_1, \dots, a_k\})$ for $k \leq n$ is the set

$$P_k = \{x \in \mathbb{R}^n : a_i^{-1}x \geq 0, i = 1, \dots, k, \, a_i^{-1}x = 0, i = k + 1, \dots, n\},$$

where a_i^{-1} denotes the i-th row of A^{-1}.

Problem 3

Prove the following variant of Farkas' lemma: Let $A \in \mathbb{R}^{m\times n}$ be a matrix and $b \in \mathbb{R}^m$ be a vector. The system $Ax \leq b$, $x \in \mathbb{R}^n$ has a solution if and only if for all $\lambda \in \mathbb{R}^m_{\geq 0}$ with $\lambda^T A = 0$ one has $\lambda^T b \geq 0$. *Hint: Use the version of Farkas' lemma in the lecture notes, Theorem 3.11*

Problem 4

Consider the vectors

$$x_1 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, x_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, x_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, x_5 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}.$$

The vector

$$v = x_1 + 3x_2 + 2x_3 + x_4 + 3x_5 = \begin{pmatrix} 15 \\ 5 \\ 31 \end{pmatrix}$$

is a conic combination of the x_i.

Write v as a conic combination using only three vectors of the x_i.
Hint: Recall the proof of Carathéodory's theorem

Problem 5

Consider the following classification problem: we are given $p_1, \dots, p_N$ points in $\mathbb{R}^d$, and each point is colored either blue or red. We want to determine if there is an hyperplane $\alpha = \{ax = b\}$ that strictly separates the blue points from the red ones (i.e. such that $ap_i > b$ for all blue points and $ap_i \leq b$ for all red points) and, in case of a positive answer, find such α. Show how to solve this problem using linear programming.

Problem 6 ($\star$)

Prove that for a finite set $X \subseteq \mathbb{R}^n$ the conic hull $\text{cone}(X)$ is closed and convex.
Hint: Use Problem 2 and Carathéodory's theorem: Let $X \subseteq \mathbb{R}^n$, then for each $x \in \text{cone}(X)$ there exists a set $\widetilde{X} \subseteq X$ of cardinality at most n such that $x \in \text{cone}(\widetilde{X})$. The vectors in $\widetilde{X}$ are linearly independent.