Problem 1
Consider the polyhedron:

\[
P = \begin{cases} 
    x_1 + 2x_2 + x_3 & \leq 5 \\
    3x_1 + x_2 + x_3 & \leq 3 \\
    x_1 & \leq 1 \\
    x_1 + x_2 & \leq 2 \\
    x_2 + x_3 & \leq 3 \\
    x_1 & \geq 0 \\
    x_1 + x_2 & \geq 0 \\
    x_2 + x_3 & \geq 0 
\end{cases}
\]

State which of the following points are vertices of \( P \): \( p_0 = (0, 0, 3) \), \( p_1 = (0, 1, 1) \), \( p_2 = (1, 4, -4) \), \( p_3 = (1/2, 3/2, 0) \), \( p_4 = (1, -1, 1) \).

Problem 2
Let \( A \in \mathbb{R}^{n \times n} \) be a non-singular matrix and let \( a_1, \ldots, a_n \in \mathbb{R}^n \) be the columns of \( A \).

i) Show that \( \text{cone}(\{a_1, \ldots, a_n\}) \) is the polyhedron \( P = \{x \in \mathbb{R}^n : A^{-1}x \geq 0\} \).

ii) Show that \( \text{cone}(\{a_1, \ldots, a_k\}) \) for \( k \leq n \) is the set

\[
P_k = \{x \in \mathbb{R}^n : a_i^{-1}x \geq 0, i = 1, \ldots, k, a_i^{-1}x = 0, i = k + 1, \ldots, n\},
\]

where \( a_i^{-1} \) denotes the \( i \)-th row of \( A^{-1} \).

Problem 3
Prove the following variant of Farkas’ lemma: Let \( A \in \mathbb{R}^{m \times n} \) be a matrix and \( b \in \mathbb{R}^m \) be a vector. The system \( Ax \leq b, x \in \mathbb{R}^n \) has a solution if and only if for all \( \lambda \in \mathbb{R}_\geq 0^m \) with \( \lambda^TA = 0 \) one has \( \lambda^Tb \geq 0 \). Hint: Use the version of Farkas’ lemma in the lecture notes, Theorem 3.11

Problem 4
Consider the vectors

\[
x_1 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, x_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, x_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, x_5 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}.
\]

The vector

\[
v = x_1 + 3x_2 + 2x_3 + x_4 + 3x_5 = \begin{pmatrix} 15 \\ 5 \\ 31 \end{pmatrix}
\]

is a conic combination of the \( x_i \).
Write $v$ as a conic combination using only three vectors of the $x_i$.

*Hint: Recall the proof of Carathéodory’s theorem*

**Problem 5**

Consider the following classification problem: we are given $p_1, \ldots, p_N$ points in $\mathbb{R}^d$, and each point is colored either blue or red. We want to determine if there is a hyperplane $\alpha = \{ax = b\}$ that strictly separates the blue points from the red ones (i.e. such that $ap_i > b$ for all blue points and $ap_i \leq b$ for all red points) and, in case of a positive answer, find such $\alpha$. Show how to solve this problem using linear programming.

**Problem 6 (⋆)**

Prove that for a finite set $X \subseteq \mathbb{R}^n$ the conic hull $\text{cone}(X)$ is closed and convex.

*Hint: Use Problem 2 and Carathéodory’s theorem: Let $X \subseteq \mathbb{R}^n$, then for each $x \in \text{cone}(X)$ there exists a set $\tilde{X} \subseteq X$ of cardinality at most $n$ such that $x \in \text{cone}(\tilde{X})$. The vectors in $\tilde{X}$ are linearly independent.*