École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand

March 17, 2017

Discrete Optimization (Spring 2017)

Assignment 4

Problem 5 can be **submitted** until March 24 12:00 noon into the right box in front of MA C1 563.
You are allowed to submit your solutions in groups of at most three students.

Problem 1

Consider the vectors

$$x_1 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, x_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, x_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, x_5 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}.$$

The vector

$$v = x_1 + 3x_2 + 2x_3 + x_4 + 3x_5 = \begin{pmatrix} 15 \\ 5 \\ 31 \end{pmatrix}$$

is a conic combination of the x_i.
Write v as a conic combination using only three vectors of the x_i.
Hint: Recall the proof of Carathéodory's theorem

Problem 2

Let (1) be a linear program in inequality standard form, i.e.

$$\max\{c^{\mathrm{T}}x \mid Ax \leq b, x \in \mathbb{R}^n\} \tag{1}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$.
Prove that there is an equivalent linear program (2) of the form

$$\min\{\tilde{c}^{\mathrm{T}}x \mid \tilde{A}x = \tilde{b}, x \geq 0, x \in \mathbb{R}^{\tilde{n}}\} \tag{2}$$

where $\tilde{A} \in \mathbb{R}^{\tilde{m} \times \tilde{n}}$, $\tilde{b} \in \mathbb{R}^{\tilde{m}}$, and $\tilde{c} \in \mathbb{R}^{\tilde{n}}$ are such that every optimal point of (1) corresponds to an optimal point of (2) and vice versa.
Linear programs of the form in (2) are said to be in *equality standard form.*

Problem 3

Determine the dual program for the following linear program:

$$\begin{array}{rrcr} \min & 3x_1 + 2x_2 - 3x_3 + 4x_4 & & \\ & 2x_1 - 2x_2 + 3x_3 + 4x_4 & \leq & 3 \\ & x_2 + 3x_3 + 4x_4 & \geq & -5 \\ & 2x_1 - 3x_2 - 7x_3 - 4x_4 & = & 2 \\ & x_1 \geq 0 & & \\ & x_4 \leq 0 & & \end{array}$$

Problem 4

Consider the following classification problem: we are given $p_1, \ldots, p_N$ points in $\mathbb{R}^d$, and each point is colored either blue or red. We want to determine if there is an hyperplane $\alpha = \{ax = b\}$ that strictly separates the blue points from the red ones (i.e. such that $ap_i > b$ for all blue points and $ap_i \leq b$ for all red points) and, in case of a positive answer, find such α. Show how to solve this problem using linear programming.

Problem 5 ($\star$)

Suppose that $A \in \mathbb{R}^{m \times n}$ has full-column rank and x^* is a solution of $Ax \leq b$. Provide an algorithm that computes an extreme point of $P = \{x : Ax \leq b\}$ in polynomial time in the dimension and the encoding length of A, b, x^*.