Prof. F. Eisenbrand EPFL - DISOPT

# Combinatorial Optimization

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## Sheet 4 **November 3, 2011**

General remark:
In order to obtain a bonus for the final grading, you may hand in written solutions to the exercises marked with a star at the beginning of the exercise session on November 15.

### Exercise 1 ($\star$)

In this exercise, we show that solving the separation problem for the perfect matching polytope is sufficient in order to solve the separation problem for the matching polytope.

Let $G = (V, E)$ be a graph and $G' = (V', E')$ a copy of it. Consider the graph $\tilde{G} = (\tilde{V}, \tilde{E})$ where $\tilde{V} := V \cup V'$ and $\tilde{E} = E \cup E' \cup \{\{v, v'\} : v \in V, v' \text{ its copy}\}$.

Let $x \in \mathbb{R}^{|E|}$ be a vector and define $\tilde{x} \in \mathbb{R}^{|\tilde{E}|}$ as $\tilde{x}(e) = \begin{cases} x(e) & \text{if } e \in E \cup E' \\ 1 - x(\delta(v)) & \text{if } e = \{v, v'\} \end{cases}$

Show the following:

(i) $\tilde{x}(e) \geq 0$ for all $e \in \tilde{E}$ if and only if $\sum_{e \in \delta(v)} x(e) \leq 1$ for all $v \in V$ and $x(e) \geq 0$ for all $e \in E$ .

(ii) Every $\tilde{y} \in \mathbb{R}^{|\tilde{E}|}$ feasible in the perfect matching polytope of $\tilde{G}$ corresponds to a feasible $y \in \mathbb{R}^{|E|}$ in the matching polytope of $G$.

(iii) If $\tilde{x}(e) \geq 0$ for all $e \in \tilde{E}$, we have $\tilde{x}(\tilde{\delta}(\tilde{U})) \geq \tilde{x}(\tilde{\delta}(W \setminus X) + \tilde{x}(\tilde{\delta}(X' \setminus W')$ for each $\tilde{U} \subseteq \tilde{V}$, where $W = \tilde{U} \cap V$, $X' = \tilde{U} \cap V'$ and $W'$, $X$ are their copies in $V'$, $V$ respectively.

(iv) If $\tilde{x} \geq 0$ and there exists $\tilde{U} \subseteq \tilde{V}$ with $|\tilde{U}|$ odd and $\tilde{x}(\tilde{\delta}(\tilde{U})) < 1$, then there exists $U \subseteq V$ with $|U|$ odd and $\sum_{e \in E[U]} x(e) > \frac{|U|-1}{2}$

**Exercise 2**

Let $E$ be a finite set and let $\mathcal{I}$ be a non-empty collection of subsets of $E$ such that $I \in \mathcal{I}$ and $J \subseteq I$ implies $J \in \mathcal{I}$. Prove that the following conditions are equivalent:

(i) if $I, J \in \mathcal{I}$ and $|J| > |I|$, then $I \cup \{e\} \in \mathcal{I}$ for some $e \in J \setminus I$;
(ii) if $I, J \in \mathcal{I}$ and $|J| = |I| + 1$, then $I \cup \{e\} \in \mathcal{I}$ for some $e \in J \setminus I$;
(iii) if $I, J \in \mathcal{I}$ and $|I \setminus J| = 1$, $|J \setminus I| = 2$, then $I \cup \{e\} \in \mathcal{I}$ for some $e \in J \setminus I$.
(iv) for all $A \subseteq E$, every maximal subset $I \subseteq A$ with $I \in \mathcal{I}$ has the same cardinality.

**Exercise 3** ($\star$)

Let $G = (V, E)$ be a graph. Let $\mathcal{I} \subseteq 2^V$ be defined as follows:
For $U \subseteq V$, we have $U \in \mathcal{I}$ if and only if there exists a matching in G that covers $U$ (and possibly other vertices).
Show that $M = (V, \mathcal{I})$ is a matroid.

**Exercise 4**
Let $E$ be a finite set that is partitioned into sets $E = E_1 \cup \ldots \cup E_r$ and define a system

$$\mathcal{I} := \{S \subset E \mid |S \cap E_j| \leq 1 \text{ for all } j = 1 \ldots r\}.$$

of independent sets. Show that $(E, \mathcal{I})$ is a matroid. What is the rank of this matroid? Give a simple description of the bases of the matroid.
*Remark:* This type of matroid is called a *partition matroid*.

**Exercise 5**
Let $G = (V, E)$ be a graph and consider the Maximum Cardinality Matching problem.

(i) Show that the set system $(E, \mathcal{I})$ with $\mathcal{I} = \{M \subseteq E : M \text{ is a matching}\}$ is not a matroid. Hence, the Greedy algorithm (with respect to unit edge weights) will not necessarily produce a maximum matching.
(ii) Show that the Greedy algorithm applied to the above set system produces a solution that is at least half the size of an optimal solution.