École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand

March 8, 2018

Discrete Optimization (Spring 2018)

Assignment 3

Problem 6 can be **submitted** until March 16 12:00 noon into the box in front of MA C1 563. You are allowed to submit your solutions in groups of at most three students.

Problem 1

Show that the recursion $T(n) = 8 \cdot T(n/2) + \theta(n^2)$, with the initial condition $T(1) = \theta(1)$, has the solution $T(n) = \theta(n^3)$.

Solution:
By a similar analysis as for the Strassen algorithm one obtains that the number of operations is:

$$T(n) = c \cdot \sum_{i=0}^{\log_2 n} 8^i \cdot \frac{n^2}{4^i} = c \cdot n^2 \sum_{i=0}^{\log_2 n} 2^i = c \cdot n^2 \cdot (\underbrace{2^{\log_2 n+1}}_{2n} - 1). \tag{1}$$

Thus $c \cdot n^3 \leq T(n) < 2 \cdot c \cdot n^3$ for $n \geq 1$, i.e. $T(n) = \theta(n^3)$.

Problem 2

Let I be an index set and $C_i \subseteq \mathbb{R}^n$ be a convex set for each $i \in I$, prove that $\cap_{i \in I} C_i$ is a convex set (Proposition 2.1 in the lecture notes).

Solution:
Let $u, v \in \cap_{i\in I} C_i$ and $\lambda \in [0, 1]$ then $u, v \in C_i$ $\forall i \in I$, and $\lambda u + (1-\lambda)v \in C_i$ by convexity of C_i for all $i \in I$. Thus $\lambda u + (1-\lambda)v \in \cap_{i\in I} C_i$, i.e. $\cap_{i\in I} C_i$ is a convex set.

Note: A direct corollary is that the set o feasible solutions of a linear program is convex, since it is an intersection of halfspaces and each halfspace is a convex set.

Problem 3

Given an extreme point v of a convex set K. Show that v cannot be written as a convex combination of other points in K.

Solution:
Since v is an "extreme point" there exists an inequality $a^T x \leq \beta$ valid for K such that $\{v\} = K \cap \{x \in \mathbb{R}^n : a^T x = \beta\}$. We prove the statement by contradiction. Without loss of generality assume that v can be written as a convex combination of two points $u, w \in K$, i.e. $v = \lambda u + (1-\lambda)w$. We obtain that

$$\beta = a^T v = a^T(\lambda u + (1-\lambda)w) = \lambda a^T u + (1-\lambda)a^T w < \lambda\beta + (1-\lambda)\beta = \beta,$$

a contradiction. We used that $a^T u < \beta$ since $a^T u \leq \beta$ ($a^T x \leq \beta$ is valid for K and $u \in K$) and $a^T u \neq \beta$ (v is the only point in K satisfying $a^T x \leq \beta$ with equality). Analogously $a^T w < \beta$.

Problem 4

Let (2) be a linear program in inequality standard form, i.e.

$$\max\{c^{\mathrm{T}} x \mid Ax \leq b, x \in \mathbb{R}^n\} \tag{2}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$.

Prove that there is an equivalent linear program (3) of the form

$$\min\{\tilde{c}^{\mathrm{T}}y \mid \tilde{A}y = \tilde{b}, y \geq 0, y \in \mathbb{R}^{\tilde{n}}\} \tag{3}$$

where $\tilde{A} \in \mathbb{R}^{\tilde{m}\times\tilde{n}}$, $\tilde{b} \in \mathbb{R}^{\tilde{m}}$, and $\tilde{c} \in \mathbb{R}^{\tilde{n}}$ are such that every optimal point of (2) corresponds to an optimal point of (3) and vice versa.
Linear programs of the form (3) are said to be in *equality standard form.*

Solution:
The transformation requires three steps:

1. Replace every variable x_j with two non-negative variables $x_j^+ := \max\{x_j, 0\}$ and $x_j^- := -\min\{x_j, 0\}$, and replace every occurrence of x_j with $(x_j^+ - x_j^-)$.
2. Replace every constraint of the form $a_{i1}x_1 + \ldots + a_{in}x_n \leq b_i$ with a constraint $a_{i1}x_1 + \ldots + a_{in} + s_i = b_i$, where s_i is a new, non-negative *slack* variable.
3. Multiply the objective function with -1 to obtain a minimization problem.

Combining these three steps, we can write the transformed linear program as

$$\begin{aligned}
\min \quad & -c^T x^+ + c^T x^- + 0^T s \\
\text{subject to} \quad & Ax^+ - Ax^- + s = b \\
& x^+ \geq 0 \\
& x^- \geq 0 \\
& s \geq 0
\end{aligned}$$

This is the desired form if we set $\tilde{c} = \begin{pmatrix}-c & c & 0\end{pmatrix}^T$ and $\tilde{A} = \begin{bmatrix}A & -A & I\end{bmatrix}$, where I denotes the $m \times m$ identity matrix.
Given a feasible solution x of the original linear program, we can find a feasible solution $\tilde{x} = \begin{pmatrix}x^+ & x^- & s\end{pmatrix}^T$ of the reformulated program by setting x^+ to the positive part of x, x^- to the negative part of x as above, and $s = b - Ax$. Note that $s \geq 0$ holds, since x is feasible, i.e. x satisfies $Ax \leq b$. Furthermore, $\tilde{A}\tilde{x} = Ax + b - Ax = b$ by construction. Thus $\tilde{x}$ is feasible and we have $\tilde{c}^T\tilde{x} = -c^T x$.
Conversely, given a feasible solution $\tilde{x} = \begin{pmatrix}x^+ & x^- & s\end{pmatrix}^T$ of the reformulated program, $x = x^+ - x^-$ defines a feasible solution of the original linear program. We have $Ax \leq \tilde{A}\tilde{x} = b$, since $s \geq 0$ holds by the feasibility of $\tilde{x}$. Note that x has objective function value $c^T x = -\tilde{c}^T\tilde{x}$.
This implies in particular that optimal solutions correspond to each other.

Problem 5

Consider the following linear program:

$$\begin{aligned}
\max \quad & a + 3b \\
\text{s.t.} \quad & a + b \leq 2 \\
& a \leq 1 \\
& -a \leq 0 \\
& -b \leq 0
\end{aligned}$$

Compute the optimal solution via vertex enumeration. Give an alternative proof/certificate that the vertex you found is an optimal solution.

Solution:
The given polyhedron has the following vertices:

$$x_1 = (0, 0)^T$$
$$x_2 = (1, 0)^T$$
$$x_3 = (1, 1)^T$$
$$x_4 = (0, 2)^T.$$

The vertex with the maximum objective value is x_4. This can also be checked easily with a drawing. An alternative certificate is the vector $\lambda^T = (3, 2)$ as in Problem 3 of Assignment 1.

Problem 6 ($\star$)

Suppose that $A \in \mathbb{R}^{m \times n}$ has full-column rank and x^* is a feasible solution to $Ax \leq b$. Provide an algorithm that computes an extreme point of $P = \{x : Ax \leq b\}$ in polynomial time in the dimension and the encoding length of A, b, x^*.

Hint: See the proof of Theorem 3.2 in the lecture notes.

Solution:
Let $A'x \leq b'$ be the subsystem of $Ax \leq b$ satisfied with equality by x^*. If $rk(A') = n$, then x^* is an extreme point. Otherwise, we will use the fact that $rk(A') < n$ to move from x^* to another point that satisfies with equality a subsystem of larger rank. This implies that, in at most n steps, we will obtain an extreme point. Let $\tilde{a}$ be a row of A that is linearly independent with the rows of A' (in other words, the matrix obtained by adding the row $\tilde{a}$ to A' has rank equal to $rk(A) + 1$). We can find a vector d such that $A'd = 0$, and $\tilde{a}d = 1$ by solving a system of equations of rank at most n. Let I be the subset of rows a_i of A such that $a_i d > 0$. Notice that I is not empty ($\tilde{a} \in I$) and the rows in I are linearly independent with the rows of A'. Let

$$\epsilon = \min_{a_i \in I} \frac{b_i - a_i x^*}{a_i d}$$

Then it is easy to verify that $x^* + \epsilon d$ is a point of P that satisfies with equality a subsystem of rank at least $rk(A') + 1$ ($A'x \leq b'$ and at least one inequality from I which is linearly independent with A'). Notice that all the operations done (solving a linear system e.g. by Gaussian elimination, finding I and ϵ) take polynomial time.