Problem 1
Show that the recursion \( T(n) = 8 \cdot T(n/2) + \theta(n^2) \), with the initial condition \( T(1) = \theta(1) \), has the solution \( T(n) = \theta(n^3) \).

Solution:
By a similar analysis as for the Strassen algorithm one obtains that the number of operations is:

\[
T(n) = c \cdot \sum_{i=0}^{\log_2 n} 8^i \cdot \frac{n^2}{4^i} = c \cdot n^2 \sum_{i=0}^{\log_2 n} 2^i = c \cdot n^2 \cdot \frac{(2^{\log_2 n+1} - 1)}{2n}.
\]  

(1)

Thus \( c \cdot n^3 \leq T(n) < 2 \cdot c \cdot n^3 \) for \( n \geq 1 \), i.e. \( T(n) = \theta(n^3) \).

Problem 2
Let \( I \) be an index set and \( C_i \subseteq \mathbb{R}^n \) be a convex set for each \( i \in I \), prove that \( \cap_{i \in I} C_i \) is a convex set (Proposition 2.1 in the lecture notes).

Solution:
Let \( u, v \in \cap_{i \in I} C_i \) and \( \lambda \in [0, 1] \) then \( u, v \in C_i \ \forall i \in I \), and \( \lambda u + (1 - \lambda)v \in C_i \) by convexity of \( C_i \) for all \( i \in I \). Thus \( \lambda u + (1 - \lambda)v \in \cap_{i \in I} C_i \), i.e. \( \cap_{i \in I} C_i \) is a convex set.

Note: A direct corollary is that the set of feasible solutions of a linear program is convex, since it is an intersection of halfspaces and each halfspace is a convex set.

Problem 3
Given an extreme point \( v \) of a convex set \( K \). Show that \( v \) cannot be written as a convex combination of other points in \( K \).

Solution:
Since \( v \) is an "extreme point" there exists an inequality \( a^T x \leq \beta \) valid for \( K \) such that \( \{v\} = K \cap \{x \in \mathbb{R}^n : a^T x = \beta\} \). We prove the statement by contradiction. Without loss of generality assume that \( v \) can be written as a convex combination of two points \( u, w \in K \), i.e. \( v = \lambda u + (1 - \lambda)w \). We obtain that

\[
\beta = a^T v = a^T (\lambda u + (1 - \lambda)w) = \lambda a^T u + (1 - \lambda) a^T w < \lambda \beta + (1 - \lambda) \beta = \beta,
\]

a contradiction. We used that \( a^T u < \beta \) since \( a^T u \leq \beta \) (\( a^T x \leq \beta \) is valid for \( K \) and \( u \in K \)) and \( a^T u \neq \beta \) (\( v \) is the only point in \( K \) satisfying \( a^T x \leq \beta \) with equality). Analogously \( a^T w < \beta \).

Problem 4
Let (2) be a linear program in inequality standard form, i.e.

\[
\max \{ c^T x \mid Ax \leq b, x \in \mathbb{R}^n \} \tag{2}
\]

where \( A \in \mathbb{R}^{m \times n} \), \( b \in \mathbb{R}^m \), and \( c \in \mathbb{R}^n \).
Prove that there is an equivalent linear program (3) of the form

\[
\min \{ \tilde{c}^T y \mid \tilde{A}y = \tilde{b}, y \geq 0, y \in \mathbb{R}^{\tilde{n}} \}
\]  

(3)

where \( \tilde{A} \in \mathbb{R}^{\tilde{m} \times \tilde{n}} \), \( \tilde{b} \in \mathbb{R}^{\tilde{m}} \), and \( \tilde{c} \in \mathbb{R}^{\tilde{n}} \) are such that every optimal point of (2) corresponds to an optimal point of (3) and vice versa.

Linear programs of the form (3) are said to be in equality standard form.

Solution:
The transformation requires three steps:

1. Replace every variable \( x_j \) with two non-negative variables \( x_j^+ := \max\{x_j, 0\} \) and \( x_j^- := -\min\{x_j, 0\} \), and replace every occurrence of \( x_j \) with \( (x_j^+ - x_j^-) \).

2. Replace every constraint of the form \( a_{i1}x_1 + \ldots + a_{in}x_n \leq b_i \) with a constraint \( a_{i1}x_1 + \ldots + a_{in} + s_i = b_i \), where \( s_i \) is a new, non-negative slack variable.

3. Multiply the objective function with \(-1\) to obtain a minimization problem.

Combining these three steps, we can write the transformed linear program as

\[
\min \quad -c^T x^+ + c^T x^- + 0^T s
\]

subject to \[
Ax^+ - Ax^- + s = b \\
x^+ \geq 0 \\
x^- \geq 0 \\
s \geq 0
\]

This is the desired form if we set \( \tilde{c} = (-c \ c \ 0)^T \) and \( \tilde{A} = [A \ -A \ I] \), where \( I \) denotes the \( m \times m \) identity matrix.

Given a feasible solution \( x \) of the original linear program, we can find a feasible solution \( \tilde{x} = (x^+ \ x^- \ s)^T \) of the reformulated program by setting \( x^+ \) to the positive part of \( x \), \( x^- \) to the negative part of \( x \) as above, and \( s = b - Ax \). Note that \( s \geq 0 \) holds, since \( x \) is feasible, i.e. \( x \) satisfies \( Ax \leq b \). Furthermore, \( \tilde{A} \tilde{x} = Ax + b - Ax = b \) by construction. Thus \( \tilde{x} \) is feasible and we have \( \tilde{c}^T \tilde{x} = -c^T x \).

Conversely, given a feasible solution \( \tilde{x} = (x^+ \ x^- \ s)^T \) of the reformulated program, \( x = x^+ - x^- \) defines a feasible solution of the original linear program. We have \( Ax \leq \tilde{A} \tilde{x} = b \), since \( s \geq 0 \) holds by the feasibility of \( \tilde{x} \). Note that \( x \) has objective function value \( c^T x = -\tilde{c}^T \tilde{x} \).

This implies in particular that optimal solutions correspond to each other.

Problem 5
Consider the following linear program:

\[
\max \quad a + 3b \\
\text{s.t.} \quad a + b \leq 2 \\
\quad \quad a \leq 1 \\
\quad \quad -a \leq 0 \\
\quad \quad -b \leq 0
\]

Compute the optimal solution via vertex enumeration. Give an alternative proof/certificate that the vertex you found is an optimal solution.
Solution:
The given polyhedron has the following vertices:
\[ x_1 = (0, 0)^T \]
\[ x_2 = (1, 0)^T \]
\[ x_3 = (1, 1)^T \]
\[ x_4 = (0, 2)^T. \]

The vertex with the maximum objective value is \( x_4 \). This can also be checked easily with a drawing. An alternative certificate is the vector \( \lambda^T = (3, 2) \) as in Problem 3 of Assignment 1.

Problem 6 (∗)
Suppose that \( A \in \mathbb{R}^{m \times n} \) has full-column rank and \( x^* \) is a feasible solution to \( Ax \leq b \). Provide an algorithm that computes an extreme point of \( P = \{x : Ax \leq b\} \) in polynomial time in the dimension and the encoding length of \( A, b, x^* \).

\textit{Hint: See the proof of Theorem 3.2 in the lecture notes.}

Solution:
Let \( A'x \leq b' \) be the subsystem of \( Ax \leq b \) satisfied with equality by \( x^* \). If \( rk(A') = n \), then \( x^* \) is an extreme point. Otherwise, we will use the fact that \( rk(A') < n \) to move from \( x^* \) to another point that satisfies with equality a subsystem of larger rank. This implies that, in at most \( n \) steps, we will obtain an extreme point. Let \( \tilde{a} \) be a row of \( A \) that is linearly independent with the rows of \( A' \) (in other words, the matrix obtained by adding the row \( \tilde{a} \) to \( A' \) has rank equal to \( rk(A') + 1 \)). We can find a vector \( d \) such that \( A'd = 0 \), and \( \tilde{a}d = 1 \) by solving a system of equations of rank at most \( n \). Let \( I \) be the subset of rows \( a_i \) of \( A \) such that \( a_id > 0 \). Notice that \( I \) is not empty (\( \tilde{a} \in I \)) and the rows in \( I \) are linearly independent with the rows of \( A' \). Let
\[ \epsilon = \min_{a_i \in I} \frac{b_i - a_ix^*}{a_id} \]

Then it is easy to verify that \( x^* + \epsilon d \) is a point of \( P \) that satisfies with equality a subsystem of rank at least \( rk(A') + 1 \) (\( A'x \leq b' \) and at least one inequality from \( I \) which is linearly independent with \( A' \)). Notice that all the operations done (solving a linear system e.g. by Gaussian elimination, finding \( I \) and \( \epsilon \)) take polynomial time.