École Polytechnique Fédérale de Lausanne
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**Combinatorial Optimization** (Fall 2016)

# Assignment 3

Deadline: October 21 10:00, into the right box in front of MA C1 563.

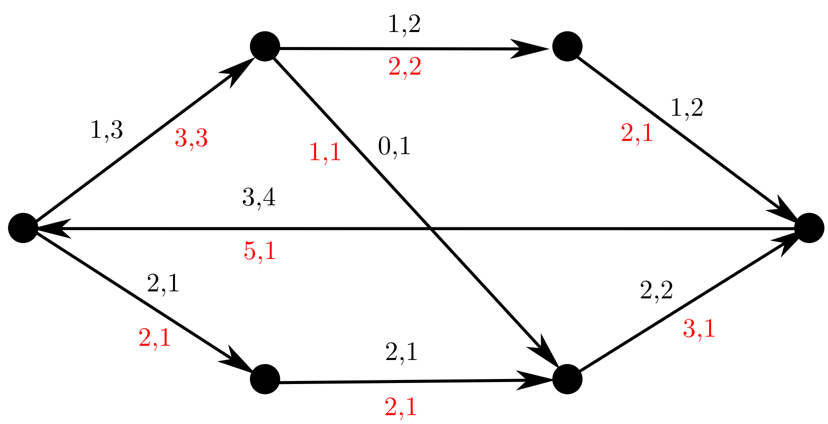
Exercises marked with a $\star$ can be handed in for bonus points.

In the exercises you might need the following lemma, which will be proved in the next lecture. It can also be proved with linear programming, using the fact that the node-arc incidence matrix of a directed graph is Totally Unimodular.
**Lemma:** Let $D = (V, A)$ be a directed graph and $c : A \to \mathbb{Z}$ be an integral capacity function, and $w : A \to \mathbb{R}$ a cost function. Then there exist a circulation $c$ of minimum cost such that $c(a)$ is integer for any $a \in A$.

### Problem 1

In the graph $D$ below, the numbers in red represent respectively the capacity $c$ and the cost $w$ of the arcs, and the numbers in black represent two circulations $f_1, f_2$. Draw the residual network $D_{f_2}$.

- Compute $\overline{f_1 - f_2}$. Verify that it is a circulation in $D_{f_2}$ and that $w'(\overline{f_1 - f_2}) = w(f_1) - w(f_2)$ (where $w'$ is the cost function of $D_{f_2}$).
- Find a directed cycle of negative cost in $D_{f_2}$. Augment $f_2$ of the minimum residual capacity along the cycle, and verify that the circulation obtained has smaller cost than $f_2$.




### Problem 2

Let $D = (V, A)$ be a directed graph and $l : A \to \mathbb{R}_+$ a length function on the edges. Given a node $s \in V$, we want to determine the shortest distance $d(s, v)$ for any $v \in V \setminus \{s\}$. Model this problem as a minimum cost circulation problem.
*Hint:* You can solve $|V| - 1$ different problems: for each $v \in V \setminus \{s\}$, transform the graph into $D_v$ adding costs and capacities (you can add extra vertices and/or arcs) so that, given the minimum cost circulation for that graph, you can obtain the shortest distance $d(s, v)$. The costs can be negative.

**Problem 3**

Construct directed graphs with integral capacities having:

1. Many different minimum cuts (for instance, exponentially many in the number of vertices) and a unique maximum flow.
2. Many maximum flows and a unique minimum cut.
3. Many maximum flows and many minimum cuts.

**Problem 4** ($\star$)

Let $G = (A \cup B, E)$ be a bipartite graph with bipartition $A, B$.

1. Let $w : E \to \mathbb{R}_+$ a function on the edges. Model the problem of finding a matching in $G$ of maximum weight as a minimum cost circulation problem.

   *Hint:* Transform the graph in a directed network with cost and capacity functions, such that each (integral) circulation corresponds to a matching. Keep in mind that you need to transform a maximization problem into a minimization problem.

2. Suppose $|A| = |B|$. Using the max-flow min-cut theorem, prove Hall's theorem: there is a perfect matching in $G$ if and only if $\forall S \subseteq A, |S| \leq |N(S)|$ (where $N(S) = \{v \in B : (u, v) \in E$ for some $u \in S\}$).