École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand
Assistant: Manuel Aprile

October 7, 2016

Combinatorial Optimization (Fall 2016)

Assignment 3

Deadline: October 21 10:00, into the right box in front of MA C1 563.

Exercises marked with a $\star$ can be handed in for bonus points.

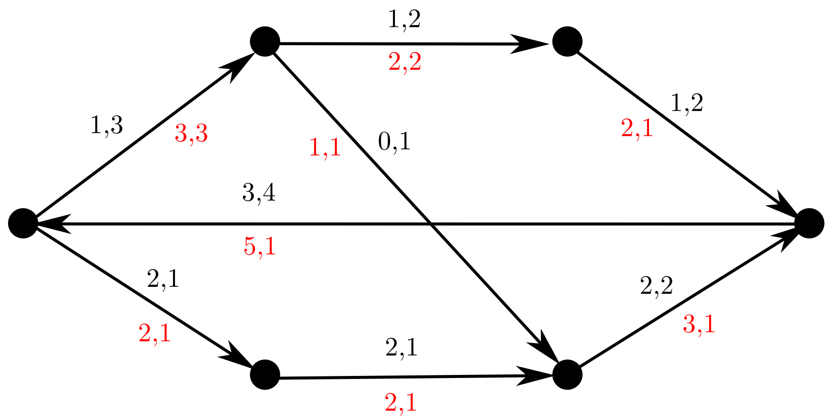
In the exercises you might need the following lemma, which will be proved in the next lecture. It can also be proved with linear programming, using the fact that the node-arc incidence matrix of a directed graph is Totally Unimodular.

Lemma: Let $D = (V, A)$ be a directed graph and $c : A \to \mathbb{Z}$ be an integral capacity function, and $w : A \to \mathbb{R}$ a cost function. Then there exist a circulation f of minimum cost such that $f(a)$ is integer for any $a \in A$.

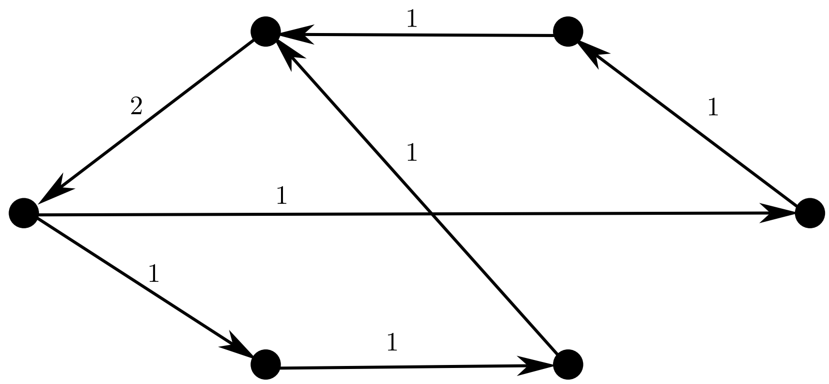
Problem 1

In the graph D below, the numbers in red represent respectively the capacity c and the cost w of the arcs, and the numbers in black represent two circulations f_1, f_2. Draw the residual network D_{f_2}.

- Compute $\overline{f_1 - f_2}$. Verify that it is a circulation in D_{f_2} and that $w'(\overline{f_1 - f_2}) = w(f_1) - w(f_2)$ (where w' is the cost function of D_{f_2}).
- Find a directed cycle of negative cost in D_{f_2}. Augment f_2 of the minimum residual capacity along the cycle, and verify that the circulation obtained has smaller cost than f_2.



Solution:
The circulation $\overline{f_1 - f_2}$ is depicted below and has cost equal to $-9 = 15 - 24$.



Problem 2

Let $D = (V, A)$ be a directed graph and $l : A \to \mathbb{R}_+$ a length function on the edges. Given a node $s \in V$, we want to determine the shortest distance $d(s, v)$ for any $v \in V \setminus \{s\}$. Model this problem as a minimum cost circulation problem.
Hint: You can solve $|V| - 1$ different problems: for each $v \in V \setminus \{s\}$, transform the graph into D_v adding costs and capacities (you can add extra vertices and/or arcs) so that, given the minimum cost circulation for that graph, you can obtain the shortest distance $d(s, v)$. The costs can be negative.

Solution:
As in the hint, we solve the problem for each vertex separately. Fix $v \in V \setminus \{s\}$. D_v has the same vertices as D and arcs $A \cup \{vs\}$ (similarly as we have seen in class, we can suppose without loss of generality that there is no vs arc in A). We give capacity 1 to all arcs. For each $a \in A$, we set the cost $w(a) = l(a)$. The cost of vs must be negative, and its absolute value big enough so that a minimum cost circulation must give value 1 to vs. For instance, if $w(vs) = -\sum_{a \in A} l(a) - 1$, the the cost of an integral circulation f is negative if and only if $f(vs) = 1$. Now, for any integral circulation with $f(vs) = 1$, by flow conservation the arcs $a \in A$ such that $f(a) = 1$ form a $s - v$ path P. The total cost of f is then $w(vs) + w(P)$, hence f is minimized when $w(P) = l(P)$ is minimum, i.e. P is the shortest $s - v$ path in D.

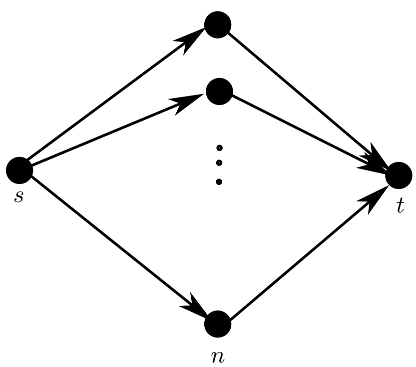
Problem 3

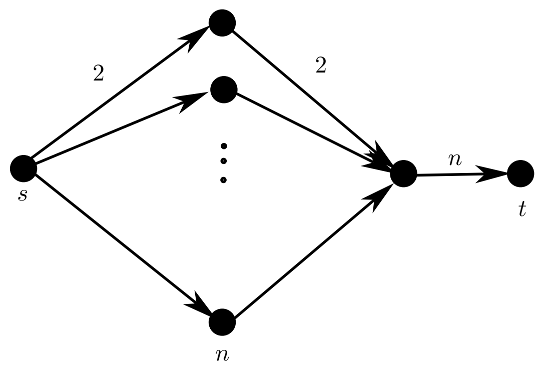
Construct directed graphs with integral capacities having:

1. Many different minimum cuts (for instance, exponentially many in the number of vertices) and a unique maximum flow.
2. Many maximum flows and a unique minimum cut.
3. Many maximum flows and many minimum cuts.

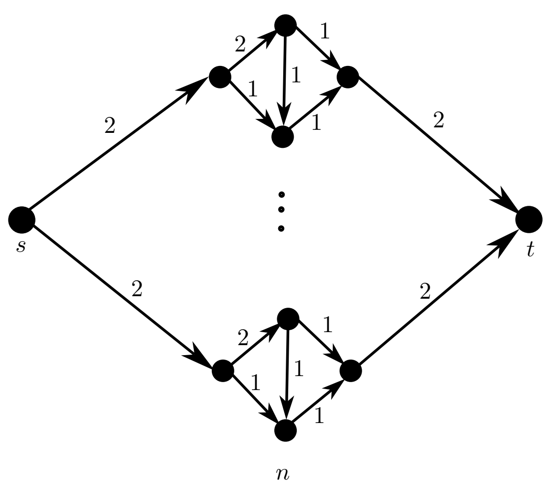
Solution:

The two graphs below satisfy 1. (2^n minimum cuts and a unique maximum flow of value n) and 2. (2^n flows of value n and a unique cut of the same value) respectively. In the first one, all capacities are 1, in the second, all capacities are 2, apart from one arc of capacity n.





The graph below satisfies 3. in that it admits 2^n maximum flows and 2^n minimum cuts. Note that it has $4n + 2$ vertices. Can you satisfy 3. on a graph of only $n + 2$ vertices?



Problem 4 ($\star$)

Let $G = (A \cup B, E)$ be a bipartite graph with bipartition A, B.

1. Let $w : E \to \mathbb{R}_+$ a function on the edges. Model the problem of finding a matching in G of maximum weight as a minimum cost circulation problem.

   Hint: Transform the graph in a directed network with cost and capacity functions, such that each (integral) circulation corresponds to a matching. Keep in mind that you need to transform a maximization problem into a minimization problem.

2. Suppose $|A| = |B|$. Using the max-flow min-cut theorem, prove Hall's theorem: there is a perfect matching in G if and only if $\forall S \subseteq A, |S| \leq |N(S)|$ (where $N(S) = \{v \in B : (u, v) \in E$ for some $u \in S\}$).

Solution:

1. Construct a network with vertex set $A \cup B \cup \{s, t\}$, and arcs: (u, v) for any $u \in A, v \in B$ such that $(u, v) \in E$; (s, u) for any $u \in A$; (v, t) for any $v \in B$; (t, s). The capacities on all arcs are 1, except for (t, s) which has capacity $|E|$ (we might also set it to infinity). The cost of any arc (u, v) with $u \in A$ and $v \in B$ is set to $-w(u, v)$. The other costs are set to 0. Now, let M be a matching of G, it is easy to build a circulation of cost $-w(M)$: for every $(u, v) \in M$ set $f(u, v) = f(s, u) = f(v, t) = 1$, and $f(t, s) = |M|$. On the other hand, given an integral circulation f, the edges (u, v) such that $f(u, v) = 1$ form a matching in G whose weight is the opposite of the cost of f. Hence the circulation of minimum cost will give us the matching of maximum weight.

2. Add two vertices s and t and the arc (s, u) and (v, t) for all $u \in A$ and $v \in B$, and orient the edges in E from A to B. Set all capacities to 1. An integral max-flow produced by the Ford-Fulkerson algorithm takes the values 0/1 and the edges between A and B with flow 1 form a matching of maximum cardinality.

   Suppose for a contradiction that the largest matching is of size $n - 1$ (where $n = |A| = |B|$), while Hall's condition is satisfied. Then the value of the max-flow is also at most $n - 1$. Let (L, R) be a cut of capacity $c \leq n - 1$, with $s \in L, R = V \setminus L \cup \{t\}$. Suppose that L contains k vertices from A and l vertices from B. Then $c = n - k + l + \bar{E} \leq n - 1$ i.e. $l + \bar{E} \leq k - 1$, where $\bar{E}$ is the number of edges between $L \setminus A$ and $R \setminus B$. By Hall's condition, $k \leq |N(L \cap A)|$ and also $\bar{E} + l \geq |N(L \cap A)|$, which gives a contradiction.

   Now suppose there exists a perfect matching. Then the value of the maximum flow is n and any cut will have capacity $\geq n$. If there is $S \subseteq A$ such that $|S| > |N(S)|$, then the cut $S \cup \{s\}$ will have capacity $n - |S| + |N(S)| < n$, a contradiction.