Problem 1
Show that the recursion \( T(n) = 8 \cdot T(n/2) + \theta(n^2) \), with the initial condition \( T(1) = \theta(1) \), has the solution \( T(n) = \theta(n^3) \).

Problem 2
Let \( I \) be an index set and \( C_i \subseteq \mathbb{R}^n \) be a convex set for each \( i \in I \), prove that \( \cap_{i \in I} C_i \) is a convex set (Proposition 3.1 in the lecture notes).

Problem 3
Given an extreme point \( v \) of a convex set \( K \). Show that \( v \) cannot be written as a convex combination of other points in \( K \).

Problem 4
Let (1) be a linear program in inequality standard form, i.e.

\[
\max \{ c^T x \mid Ax \leq b, x \in \mathbb{R}^n \}
\]

where \( A \in \mathbb{R}^{m \times n} \), \( b \in \mathbb{R}^m \), and \( c \in \mathbb{R}^n \).

Prove that there is an equivalent linear program (2) of the form

\[
\min \{ \bar{c}^T y \mid \bar{A}y = \bar{b}, y \geq 0, y \in \mathbb{R}^{\bar{n}} \}
\]

where \( \bar{A} \in \mathbb{R}^{\bar{m} \times \bar{n}} \), \( \bar{b} \in \mathbb{R}^{\bar{m}} \), and \( \bar{c} \in \mathbb{R}^{\bar{n}} \) are such that every optimal point of (1) corresponds to an optimal point of (2) and vice versa.

Linear programs of the form (2) are said to be in equality standard form.

Problem 5
Consider the following linear program:

\[
\max \quad a + 3b \\
\text{s.t.} \quad a + b \leq 2 \\
\quad \quad \quad \quad \quad a \leq 1 \\
\quad \quad \quad \quad \quad -a \leq 0 \\
\quad \quad \quad \quad \quad -b \leq 0
\]

Compute the optimal solution via vertex enumeration. Give an alternative proof/certificate that the vertex you found is an optimal solution.

Problem 6 (⋆)
Suppose that \( A \in \mathbb{R}^{m \times n} \) has full-column rank and \( x^* \) is a feasible solution to \( Ax \leq b \). Provide an algorithm that computes an extreme point of \( P = \{ x : Ax \leq b \} \) in polynomial time in the dimension and the encoding length of \( A, b, x^* \).

Hint: See the proof of Theorem 3.2 in the lecture notes.