École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand

March 9, 2017

Discrete Optimization (Spring 2017)

Assignment 3

Problem 5 can be **submitted** until March 17 12:00 noon into the right box in front of MA C1 563.
You are allowed to submit your solutions in groups of at most three students.

Problem 1

Let I be an index set and $C_i \subseteq \mathbb{R}^n$ be a convex set for each $i \in I$, prove that $\cap_{i\in I} C_i$ is a convex set (Proposition 2.1 in the lecture notes).

Problem 2

Given an "extreme point" v of a convex set K. Show that v cannot be written as a convex combination of other points in K.

Problem 3

Let $A \in \mathbb{R}^{n\times n}$ be a non-singular matrix and let $a_1, \dots, a_n \in \mathbb{R}^n$ be the columns of A.

i) Show that $\text{cone}(\{a_1, \dots, a_n\})$ is the polyhedron $P = \{y \in \mathbb{R}^n \colon A^{-1}y \geq 0\}$.

ii) Show that $\text{cone}(\{a_1, \dots, a_k\})$ for $k \leq n$ is the set

$$P_k = \{y \in \mathbb{R}^n \colon a_i^{-1} y \geq 0, i = 1, \dots, k,\ a_i^{-1} y = 0, i = k+1, \dots, n\},$$

where a_i^{-1} denotes the i-th row of A^{-1}.

Problem 4

Prove that for a finite set $X \subseteq \mathbb{R}^n$ the conic hull $\text{cone}(X)$ is closed and convex. Use the previous exercise and Carathéodory's theorem: Let $X \subseteq \mathbb{R}^n$, then for each $x \in \text{cone}(X)$ there exists a set $\widetilde{X} \subseteq X$ of cardinality at most n such that $x \in \text{cone}(\widetilde{X})$. The vectors in $\widetilde{X}$ are linearly independent.

Problem 5 ($\star$)

Prove the following variant of Farkas' lemma: Let $A \in \mathbb{R}^{m\times n}$ be a matrix and $b \in \mathbb{R}^m$ be a vector. The system $Ax \leq b$, $x \in \mathbb{R}^n$ has a solution if and only if for all $\lambda \in \mathbb{R}^m_{\geq 0}$ with $\lambda^T A = 0$ one has $\lambda^T b \geq 0$. *Hint: Use the version of Farkas' lemma proven in class.*