Exercise 1

1. Show that the unit simplex \( \Delta = \text{conv}\{0, e_1, \ldots, e_n\} \subset \mathbb{R}^n \), where \( e_j \) are the standard unit vectors, has volume \( \frac{1}{n!} \).

2. Show that the volume of the simplex \( \text{conv}\{v_0, v_1, \ldots, v_n\} \subset \mathbb{R}^n \) is \( \frac{|\det(v_1-v_0, \ldots, v_n-v_0)|}{n!} \).

Exercise 2 (⋆)
Describe an algorithm that given as input
- an integral polyhedron \( P = \{x \in \mathbb{R}^n \mid Ax \leq b\} \), where \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \),
- an objective function vector \( c \in \mathbb{Z}^n \),
- the optimal objective function value \( z = \max\{c^T x \mid x \in P\} \), and
- a feasible point \( x^* \in P \) with \( z - \frac{1}{2} \leq c^T x^* \leq z \)
computes an inclusion-wise minimal optimal face \( F = \{x \in \mathbb{R}^n \mid A'x = b'\} \) of \( P \). The running time of your algorithm shall be bounded by a polynomial in \( n \) and \( m \).

Exercise 3
Reduce the problem of finding a maximum weight matching to the problem of finding a maximum weight perfect matching. That is, find a way to transform, in polynomial time, any graph \( G \) with edge weights \( w \) into a graph \( G' \) with edge weights \( w' \) so that given a maximum weight perfect matching \( M' \) of \( G' \), one can easily deduce a maximum weight matching \( M \) of \( G \).

Exercise 4 (⋆)
Let \( \mathcal{F} \subseteq \{0, 1\}^n \) and assume access to an oracle that given \( \bar{x} \in \mathcal{F} \) and \( \bar{c} \in \mathbb{Z}^n \) either
- asserts that \( \bar{x} \in \mathcal{F} \) maximizes \( \bar{c}^T x \) over \( x \in \mathcal{F} \), or
- returns an \( x \in \mathcal{F} \) such that \( \bar{c}^T x > \bar{c}^T \bar{x} \).

Describe an algorithm that given \( c \in \mathbb{Z}^n \) and an initial feasible solution \( x \in \mathcal{F} \) computes an optimal solution of \( \max\{c^T x \mid x \in \mathcal{F}\} \) with a running time that is bounded by a polynomial in \( n \) and \( \log|c|_{\infty} \).