École Polytechnique Fédérale de Lausanne
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September 30, 2016

Combinatorial Optimization (Fall 2016)

Assignment 2

Deadline: October 7 10:00, into the right box in front of MA C1 563.

Exercises marked with a $\star$ can be handed in for bonus points.

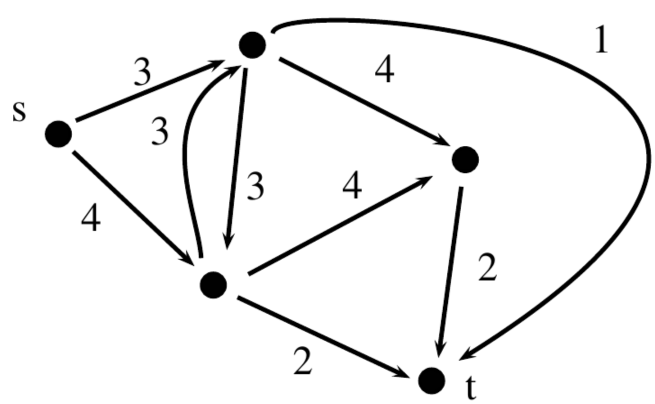
Recall the max-flow algorithm seen in class.

Given: a digraph $G(V, A)$ with capacities $u : A \to Q_{\geq 0}$, and a pair of distinct vertices $s, t \in V$.

- Set $f = 0$ and construct the auxiliary graph D_f.
- While there is an oriented path between s and t in D_f:
  - Augment f to f' (*augmenting step*).
  - Set $f = f'$.
  - Construct the auxiliary graph D_f.
- Output f.

Problem 1

Determine with the max-flow algorithm an $s-t$ flow of maximum value and an $s-t$ cut of minimum capacity in the following graph (where the numbers at the arcs give the capacities):



Solution:
A maximum $s - t$ flow has value 5.

Problem 2

Let $D(V, A)$ be a directed graph. Using the Max-flow Min-cut Theorem, prove the following version of Menger's Theorem: the maximum number of arc-disjoint $s - t$ paths is equal to the minimum size of a subset $B \subset A$ whose removal disconnects s and t.

Solution:
We assign capacity 1 to every arc. Let f be a maximum flow and let $\delta^+(U)$ be a minimum cut, (so $s \in U, t \notin U$). Clearly deleting the arcs of any $s-t$ cut disconnects s and t. Hence, if $B \subset A$ is the subset of minimum size whose removal disconnects s, t, we have $|B| \leq |\delta^+(U)|$. Furthermore the flow f (which can be chosen to be integral, i.e. it has 0/1 values) yields a set of directed paths from s to t that are pairwise edge-disjoint (since any arc has at most one unit of flow assigned to it and thereby appears in at most one $s-t$ path). So the value of the flow is at most the maximum number of arc-disjoint $s-t$ paths, which we denote by N_P. By the Max-flow Min-cut Theorem, we get $|B| \leq N_P$. To conclude, we only need to show that $|B| \geq N_P$, but this is clear since we must remove at least one arc from each of the arc-disjoint $s-t$ paths.

Problem 3

Let $G = (V, E)$ be a graph and let $c : E \to \mathbb{R}^+$ be a capacity function. Let K be the complete graph on V. For each edge st of K, let $w(st)$ be the minimum capacity of any $s-t$ cut in G. (An $s-t$ cut is any subset $\delta(U)$ with $s \in U, t \notin U$.)
Let T be a spanning tree in K of maximum total weight with respect to the function w. Prove that for all $s, t \in V$, $w(st)$ is equal to the minimum weight of the edges of T in the unique $s-t$ path in T. (Hint: use an exercise from the previous assignment.)

Solution:
We use a result from the previous assignment, exercise 4. The hypotheses of the two exercises are very similar as well as the thesis, we only need to show that the function w has the property that the function l has, i.e. for any $r, s, t \in V$, $w(st) \geq \min\{w(sr), w(rt)\}$. Assume by contradiction that there are $r, s, t \in V$ such that $w(st) < \min\{w(sr), w(rt)\}$. Let U be the minimum $s-t$ cut. If $r \notin U$, then U is also a $s-r$ cut, which is a contradiction. Hence $r \in U$. But then U is an $r-t$ cut, again a contradiction.

Problem 4 ($\star$)

Enthusiastic celebration of a sunny day at a prominent Swiss university has resulted in the arrival at the university's medical clinic of 169 students in need of emergency treatment. Each of the 169 students requires a transfusion of one unit of whole blood. The clinic has supplies of 170 units of whole blood. The number of units of blood available in each of the four major blood groups and the distribution of patients among the groups is summarized below.

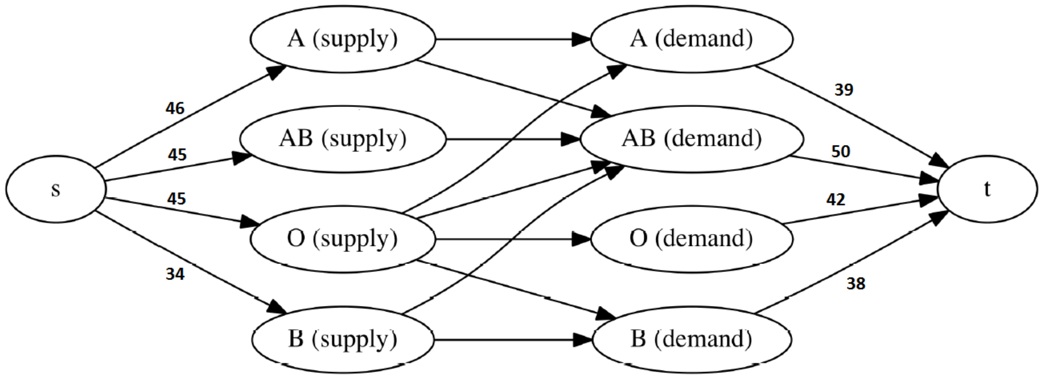
Blood type	*A*	*B*	*O*	*AB*
Supply	46	34	45	45
Demand	39	38	42	50

Type A patients can only receive type A or O; type B patients can receive only type B or O; type O patients can receive only type O; and type AB patients can receive any of the four types.

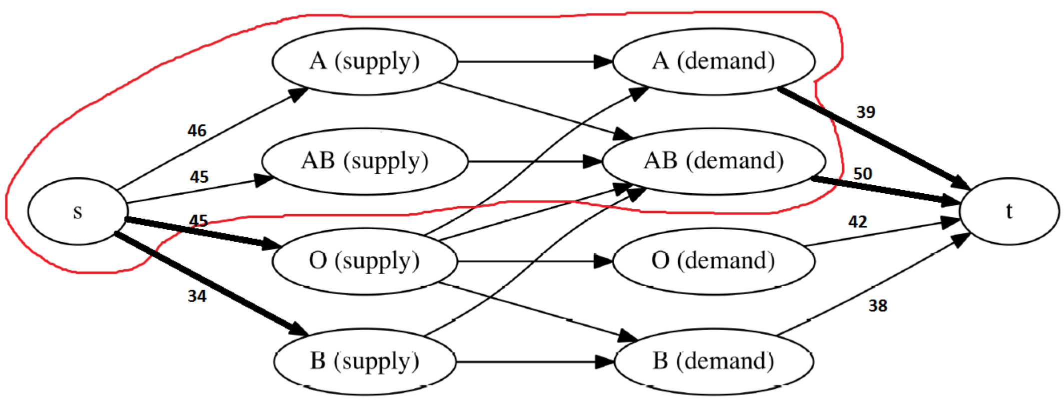
1. Give a max flow formulation that determines a distribution that satisfies the demands of a maximum number of patients. You should draw a directed graph with edge capacities such that a feasible flow corresponds to a feasible choice for the transfusion.

2. Find a cut in your graph of value smaller than 169. Use it to give an explanation of why not all of the patients can receive blood from the available supply. Your explanation should be understandable to the hospital administrators who have no knowledge of network flow theory.

Solution:
We formulate the problem as follows: there are two nodes for each blood type, representing the supply and the demand, and there is a source node s and a sink node t. The arcs represent which



type of blood in the supply can be donated to which group. The capacities represent the supply or the demand of a blood type. Note that in the edges between a supply and a demand node the capacity is set to ∞. This is an arbitrary choice. On one hand, this represents the fact that we don't constrain the amount of blood that can be given to a group. On the other hand, this will ensure that the minimum cut will not cross any of these arcs, which makes easier to look for a small cut in the graph. To find such a cut we can use the algorithm seen in class to compute the maximum flow (of value 168) and then use the technique given in the proof of Theorem 2.2 to find a cut of the same capacity.



The cut gives an intuition of why we cannot satisfy the totality of the demand. The problem lies in the blood types O and B, which are out of the cut. This means that the flow from s to O, B fills all the capacity, but it is still not enough. In fact, the demand for O is 42 units, that can only come from the supply of O, which has 45 units. On the other hand, the demand for B is 38 and the supply of B blood is only 34, hence 4 units should come from O supply, but there are only 3 left. Alternatively, we could have come up with the latter explanation and use the intuition behind it to derive the cut.