École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand
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September 30, 2016

Combinatorial Optimization (Fall 2016)

Assignment 2

Deadline: October 7 10:00, into the right box in front of MA C1 563.

Exercises marked with a $\star$ can be handed in for bonus points.

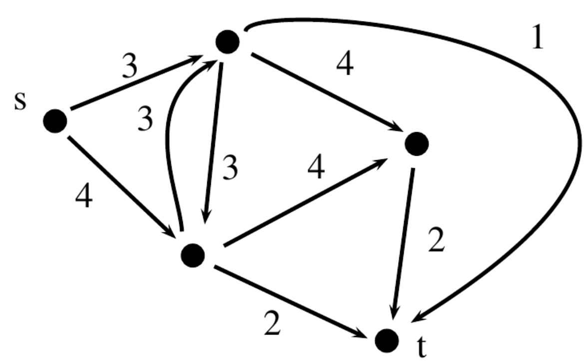
Recall the max-flow algorithm seen in class.

Given: a digraph $G(V, A)$ with capacities $u : A \to Q_{\geq 0}$, and a pair of distinct vertices $s, t \in V$.

- Set $f = 0$ and construct the auxiliary graph D_f.
- While there is an oriented path between s and t in D_f:
  - Augment f to f' (*augmenting step*).
  - Set $f = f'$.
  - Construct the auxiliary graph D_f.
- Output f.

Problem 1

Determine with the max-flow algorithm an $s-t$ flow of maximum value and an $s-t$ cut of minimum capacity in the following graph (where the numbers at the arcs give the capacities):



Problem 2

Let $D(V, A)$ be a directed graph. Using the Max-flow Min-cut Theorem, prove the following version of Menger's Theorem: the maximum number of arc-disjoint $s - t$ paths is equal to the minimum size of a subset $B \subset A$ whose removal disconnects s and t.

Problem 3

Let $G = (V, E)$ be a graph and let $c : E \to \mathbb{R}^+$ be a capacity function. Let K be the complete graph on V. For each edge st of K, let $w(st)$ be the minimum capacity of any $s - t$ cut in G. (An $s - t$ cut is any subset $\delta(U)$ with $s \in U, t \notin U$.)

Let T be a spanning tree in K of maximum total weight with respect to the function w. Prove that for all $s, t \in V$, $w(st)$ is equal to the minimum weight of the edges of T in the unique $s - t$ path in T. (Hint: use an exercise from the previous assignment.)

Problem 4 ($\star$)

Enthusiastic celebration of a sunny day at a prominent Swiss university has resulted in the arrival at the university's medical clinic of 169 students in need of emergency treatment. Each of the 169 students requires a transfusion of one unit of whole blood. The clinic has supplies of 170 units of whole blood. The number of units of blood available in each of the four major blood groups and the distribution of patients among the groups is summarized below.

Bloodtype	*A*	*B*	*O*	*AB*
Supply	46	34	45	45
Demand	39	38	42	50

Type A patients can only receive type A or O; type B patients can receive only type B or O; type O patients can receive only type O; and type AB patients can receive any of the four types.

1. Give a max flow formulation that determines a distribution that satisfies the demands of a maximum number of patients. You should draw a directed graph with edge capacities such that a feasible flow corresponds to a feasible choice for the transfusion.

2. Find a cut in your graph of value smaller than 169. Use it to give an explanation of why not all of the patients can receive blood from the available supply. Your explanation should be understandable to the hospital administrators who have no knowledge of network flow theory.