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# Combinatorial Optimization

Fall 2010

Assignment Sheet 2

**Exercise 1** $(*)$
Let $A \in \mathbb{Z}^{m\times n}$ be of full row-rank and $b \in \mathbb{Z}^m$. A matrix $U \in \mathbb{Z}^{n\times n}$ is *unimodular* if $\det(U) = \pm 1$. Show the following:

1. If $U \in \mathbb{Z}^{n\times n}$ is unimodular then $Ax = b$ has an integral solution $x^* \in \mathbb{Z}^n$ if and only if $A \cdot Ux = b$ has and integral solution $y^* \in \mathbb{Z}^n$.

2. Suppose $A'$ emerges from $A$ by adding an integer multiple of one column to **another** column. Show that $A' = A \cdot U$ with a unimodular matrix $U \in \mathbb{Z}^{n\times n}$.

3. Suppose $A'$ emerges from $A$ by swapping two columns. Show that $A' = A \cdot U$ with a unimodular matrix $U \in \mathbb{Z}^{n\times n}$.

4. Show that there exists a unimodular matrix $U \in \mathbb{Z}^{n\times n}$ such that $A \cdot U$ is a matrix that has only one nonzero entry in the first row; namely the first entry in the first row.

5. Show that there exists a unimodular transformation $U \in \mathbb{Z}^{n\times n}$ such that $A \cdot U = [B|0]$, where $B \in \mathbb{Z}^{m\times m}$ is lower-triangular, non-negative and the unique row-maximum of row $i$ of $B$ is $B(i,i)$.

   *Remark: $B$ is called the Hermite-Normal-Form of $A$.*

**Exercise 2**
Let $G = (V,E)$ and $w \in \mathbb{Z}^{|E|}$ be a counter-example to the feasibility of the inequalities (*) from the lecture with $|V| + |E|$ minimal. Show that $G$ is connected.

**Exercise 3**
Let $G = (V,E)$ be connected and $w \in \mathbb{Z}^{|E|}_{\geq 1}$ be integral positive edge-weights. Denote the set of maximum weight matchings of $G$ by $\mathcal{M}(w)$ and the maximum weight of a matching by $\mu_w$. Suppose further that the following holds: Each $M \in \mathcal{M}(w)$ satisfies $|M| = \lfloor |V|/2 \rfloor$ and $|V|$ is odd. Let $w'$ be defined by $w'(e) = w(e) - 1$ for all $e \in E$.

Show the following: $\mu_{w'} = \mu_w - \lfloor |V|/2 \rfloor$.

*Hint: Consider max-weight matchings $M'$ and $M$ w.r.t. $w'$ and $w$ respectively with $|M'|$ maximal. If $|M'| < \lfloor |V|/2 \rfloor$, then look at the symmetric difference $M' \triangle M$ and derive a contradiction.*

**Exercise 4 ($*$)**
Consider a family $S_1, \ldots, S_n$ of subsets $\{1, \ldots, m\}$. The *set-covering problem* is to choose a smallest number of these sets whose union is $\{1, \ldots, m\}$. This is modeled in the following integer linear program

$$\begin{aligned} &\min \textstyle\sum_{j=1}^{n} x_j \\ &Ax \geq \mathbf{1} \\ &x \geq 0, \text{ integral} \end{aligned}$$

where $A \in \{0,1\}^{m \times n}$ is the matrix $A(i,j) = 1$ if $i \in S_j$ and $A(i,j) = 0$ otherwise.

The goal of this exercise is to see that things are not as nice here as in the case of maximum weight matchings. We cannot expect to prove optimality of an integral solution by providing an optimal dual solution.

1. Provide an example where the linear program (integrality-constrained ignored) has a strictly smaller solution than the integer program.
2. Let $x^*$ be an optimal solution to the linear program. We are now selecting some sets in $S_1, \ldots, S_n$ at random: If $x_j^* \geq 1$, then select set $S_j$. Otherwise select $S_j$ with probability $x_j^*$. What is the expected number of selected sets?
3. Show that the probability that a particular element $i \in \{1, \ldots, m\}$ is not covered is bounded by $(1 - 1/n)^n \leq e^{-1}$.
4. If this complete rounding procedure is repeated $k$-times, then the probability that $i$ is not covered in any round is bounded by $e^{-k}$.
5. If $OPT$ denotes the optimum value of the integer linear program and $OPT_f$ denotes the optimum value of the linear program, then conclude that $OPT/OPT_f = O(\log m)$.
6. Can you find a class of set-covering problems with $OPT/OPT_f = \Omega(\log m)$?