École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand
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September 23, 2016

Combinatorial Optimization (Fall 2016)

Assignment 1

Deadline: September 30 10:00, into the right box in front of MA C1 563.

Exercises marked with a $\star$ can be handed in for bonus points. Due date is September 30.

Problem 1

Complete the proof of Theorem 1.1 seen in class. In particular, given the conditions:

i) G is a tree.

iii) G is connected, and the removal of any edge disconnects G.

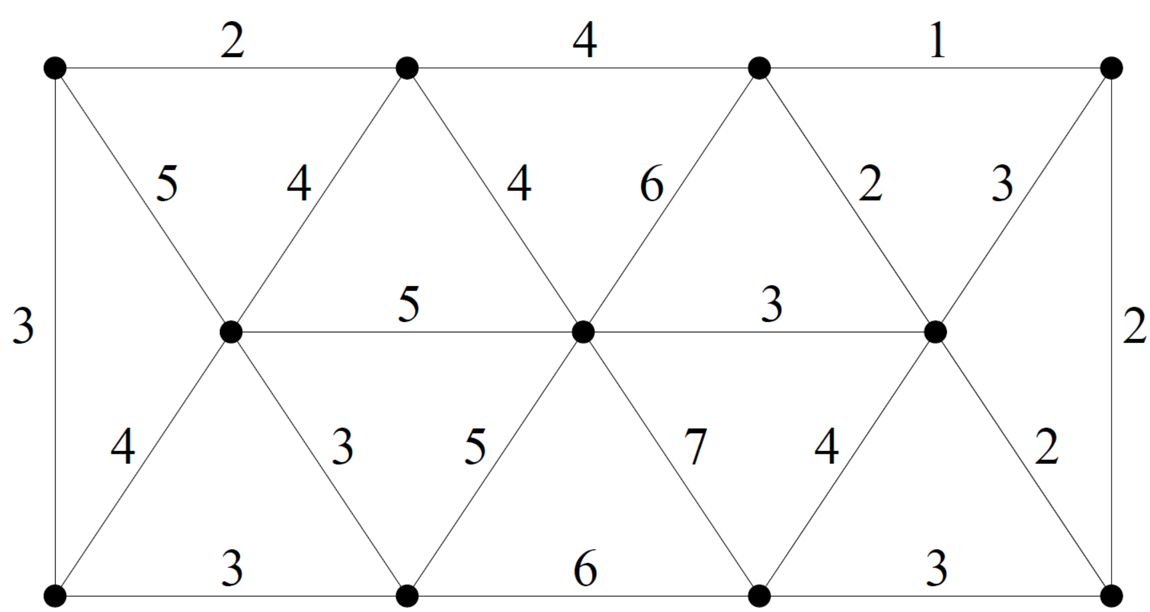
iv) G does not contain a cycle and the addition of one edge creates a cycle.

v) G is connected and $|E| = |V| - 1$.

Prove that iii) implies iv), iv) implies v) and v) implies i).

Problem 2

Find, both with the Dijkstra-Prim algorithm (Algorithm 1.1) and with Kruskal's algorithm (Algorithm 1.2), a spanning tree of minimum length in the graph below.



Problem 3

Let $G = (V, E)$ be a graph and $l : E \to \mathbb{R}$ be a length function. Call a forest $F \subset E$ *good* if $l(F') \geq l(F)$ for each forest F' satisfying $|F'| = |F|$.

Let F be a good forest and e be an edge not in F such that $F \cup \{e\}$ is a forest and such that (among all such e) $l(e)$ is as small as possible. Show that $F \cup \{e\}$ is good again.

Problem 4

Let $G = (V, E)$ be a complete graph and let $l : E \to \mathbb{R}^+$ be a length function satisfying $l(uw) \geq \min\{l(uv), l(vw)\}$ for all distinct $u, v, w \in V$. Let T be a longest spanning tree in G. Show that for all $u, w \in V$, $l(uw)$ is equal to the minimum length of the edges in the unique uw path in T.

Problem 5 ($\star$)

Let $G = (V, E)$ be a graph and $s : E \to \mathbb{R}$ be a *strength* function. For any path P, the *reliability* of P is defined as the minimum strength of the edges occurring in P. The *reliability* $r_G(u, v)$ of two vertices u, v is equal to the maximum reliability of P, where P ranges over all paths from u to v. Let T be a spanning tree of maximum strength, i.e. with $\sum_{e \in E(T)} s(e)$ as large as possible. Prove that T has the same reliability of G for any pair of vertices, that is:

$$r_T(u, v) = r_G(u, v) \;\; \forall\, u, v \in V.$$