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Discussions from: September 27, 2010

Combinatorial Optimization

Fall 2010

Assignment Sheet 1

Exercise 1

Show that the dual of the linear program $\max\{c^T x\colon x \in \mathbb{R}^n,\ Ax \leq b,\ x \geq 0\}$ for $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ can be interpreted as the linear program $\min\{b^T y\colon y \in \mathbb{R}^m,\ A^T y \geq c,\ y \geq 0\}$.

Hint: Understand the primal as $\max\{c^T x\colon x \in \mathbb{R}^n,\ \binom{A}{-I} x \leq \binom{b}{0}\}$*, and re-interpret the dual of this LP.*

Exercise 2

We could not finish the proof the following theorem during the last lecture.

Theorem. *A non-empty set* $F \subseteq \mathbb{R}^n$ *is a face of* $P = \{x \in \mathbb{R}^n\colon Ax \leq b\}$ *if and only if* $F = \{x \in P\colon A'x = b'\}$ *for a subset* $A'x \leq b'$ *of* $Ax \leq b$.

Notice that this sub-system $A'x \leq b'$ could be "empty". Here is the proof. However, a few details are missing that you should fill in.

Proof. Suppose that $F = \{x \in P\colon A'x = b'\}$. Consider the vector $c = 1^T A'$ and $\delta = 1^T b'$. The inequality $c^T x \leq \delta$ is valid for P. **It is satisfied with equality by each $\mathbf{x} \in \mathbf{F}$.** If $x' \in P \setminus F$, then there exists an inequality $a^T x \leq \beta$ of $A'x \leq b'$ such that $a^T x' < \beta$ **and consequently $\mathbf{c^T x'} < \delta$.** This shows $\{x \in P\colon A'x = b'\} = \{x \in P\colon c^T x = \delta\}$ and thus that F is a face.

On the other hand, if F is a face, then there exists a valid inequality $c^T x \leq \delta$ of P such that $F = \{x \in P\colon c^T x = \delta\}$. If $c = 0$, then clearly $F = P$ and the assertion follows. If $c \neq 0$, then since F is non-empty, linear programming duality implies

$$\delta = \max\{c^T x\colon Ax \leq b\} = \min\{b^T \lambda\colon A^T \lambda = c, \lambda \geq 0\}.$$

Thus there exists a $\lambda^* \in \mathbb{R}^m_{\geq 0}$ such that $c = \lambda^{*T} A$ and $\delta = \lambda^{*T} b$. Let $A'x \leq b'$ be the subsystem of $Ax \leq b$ which corresponds to strictly positive entries of λ^*. **One has $\mathbf{F} = \{\mathbf{x} \in \mathbf{P}\colon \mathbf{A'x} = \mathbf{b'}\}$.** □

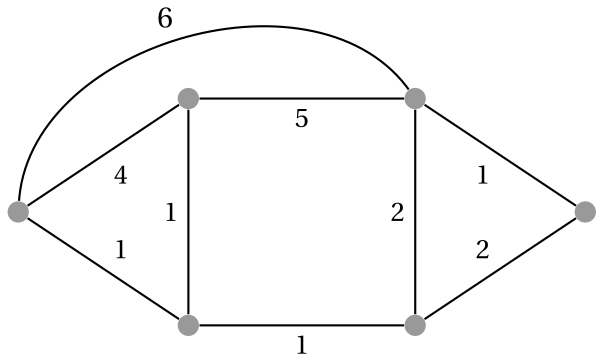
Provide a formal proof of the claims in **boldface**.

Exercise 3

Determine a maximum weight matching of the graph below. Provide of proof of optimality by determining a feasible dual solution to the linear program

$$
\begin{array}{rrcl}
\max & \sum_{e\in E} w(e)x(e) & & \\
v\in V: & \sum_{e\in\delta(v)} x(e) & \leq & 1 \\
\substack{U\subseteq V\\ |U| \text{ odd}}: & \sum_{e\in E(U)} x(e) & \leq & \lfloor |U|/2 \rfloor \\
e\in E: & 0 & \leq & x(e)
\end{array}
$$

whose objective value coincides with the weight of your matching.



Exercise 4 (⋆)
Show the following: A face F of $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is inclusion-wise minimal if and only if it is of the form $F = \{x \in \mathbb{R}^n \mid A'x = b'\}$ for some subsystem $A'x \leq b'$ of $Ax \leq b$.