Exercise 1
Show that the dual of the linear program \( \text{max}\{c^T x : x \in \mathbb{R}^n, \ A x \leq b, \ x \geq 0\} \) for \( A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \) and \( c \in \mathbb{R}^n \) can be interpreted as the linear program \( \text{min}\{b^T y : y \in \mathbb{R}^m, \ A^T y \geq c, \ y \geq 0\} \).

Hint: Understand the primal as \( \text{max}\{c^T x : x \in \mathbb{R}^n, (A - I) x \leq (b_0)\} \), and re-interpret the dual of this LP.

Exercise 2
We could not finish the proof the following theorem during the last lecture.

**Theorem.** A non-empty set \( F \subseteq \mathbb{R}^n \) is a face of \( P = \{x \in \mathbb{R}^n : A x \leq b\} \) if and only if \( F = \{x \in P : A' x = b'\} \) for a subset \( A' x \leq b' \) of \( A x \leq b \).

Notice that this sub-system \( A' x \leq b' \) could be “empty”. Here is the proof. However, a few details are missing that you should fill in.

**Proof.** Suppose that \( F = \{x \in P : A' x = b'\} \). Consider the vector \( c = 1^T A' \) and \( \delta = 1^T b' \). The inequality \( c^T x \leq \delta \) is valid for \( P \). It is satisfied with equality by each \( x \in F \). If \( x' \in P \setminus F \), then there exists an inequality \( a^T x \leq \beta \) of \( A' x \leq b' \) such that \( a^T x' < \beta \) and consequently \( c^T x' < \delta \). This shows \( \{x \in P : A' x = b'\} = \{x \in P : c^T x = \delta\} \) and thus that \( F \) is a face.

On the other hand, if \( F \) is a face, then there exists a valid inequality \( c^T x \leq \delta \) of \( P \) such that \( F = \{x \in P : c^T x = \delta\} \). If \( c = 0 \), then clearly \( F = P \) and the assertion follows. If \( c \neq 0 \), then since \( F \) is non-empty, linear programming duality implies

\[
\delta = \max\{c^T x : A x \leq b\} = \min\{b^T \lambda : A^T \lambda = c, \lambda \geq 0\}.
\]

Thus there exists a \( \lambda^* \in \mathbb{R}^m_{\geq 0} \) such that \( c = \lambda^*^T A \) and \( \delta = \lambda^*^T b \). Let \( A' x \leq b' \) be the subsystem of \( A x \leq b \) which corresponds to strictly positive entries of \( \lambda^* \). **One has** \( F = \{x \in P : A' x = b'\} \). \( \square \)

Provide a formal proof of the claims in **boldface**.

Exercise 3
Determine a maximum weight matching of the graph below. Provide of proof of optimality.
by determining a feasible dual solution to the linear program

\[
\begin{align*}
\max & \quad \sum_{e \in E} w(e) x(e) \\
\text{subject to} & \quad \sum_{e \in \delta(v)} x(e) \leq 1 \\
& \quad \sum_{e \in E(U)} x(e) \leq \lfloor |U|/2 \rfloor \\
& \quad 0 \leq x(e)
\end{align*}
\]

whose objective value coincides with the weight of your matching.

Exercise 4 (*)

Show the following: A face \( F \) of \( P = \{ x \in \mathbb{R}^n : Ax \leq b \} \) is inclusion-wise minimal if and only if it is of the form \( F = \{ x \in \mathbb{R}^n : A'x = b' \} \) for some subsystem \( A'x \leq b' \) of \( Ax \leq b \).